Cayley – Hamilton theorem :

Every square matrix satisfies its own characteristic equation.

Uses : To calculate

(i) the positive integral powers

(ii) the inverse of a square matrix A

Problems :

① If $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, write A^2 interms of A and I, using Cayley – Hamilton theorem.

Soln: The characteristic equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(5-\lambda) = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

By Cayley – Hamilton theorem,

$$A^2 - 6A + 5 = 0$$

$$\boxed{A^2 = 6A - 5I}$$

② Using Cayley – Hamilton theorem find A^4 and A^{-1} when $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln: $\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$

$C_1 = 2+2+2 = 6$

$C_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 8$

$C_3 = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 3$

$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

By Cayley-Hamilton theorem,

$A^3 - 6A^2 + 8A - 3I = 0 \rightarrow (1)$

To find A^4:

Multiply by A on both Sides, of (1),

$A^4 - 6A^3 + 8A^2 - 3A = 0$

$A^4 = 6A^3 - 8A^2 + 3A$

$= 6(6A^2 - 8A + 3I) - 8A^2 + 3A$ (using (1))

$= 36A^2 - 48A + 18I - 8A^2 + 3A$

$= 28A^2 - 45A + 18I \rightarrow (2)$

$A^2 = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$

subs A^2, A in (2),

$A^4 = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$

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To find A^{-1} :

Multiply by A^{-1} on both sides of 1.

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$A^{-1} = \frac{1}{3}\left[A^2 - 6A + 8I\right]$$

$$= \frac{1}{3}\left\{\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= \frac{1}{3}\left\{\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}\right\}$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

3 Verify that $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ satisfies its own characteristic equation.

Solution: $\lambda^2 - C_1\lambda + C_2 = 0$

$$C_1 = 1 - 1 = 0$$

$$C_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$$

$$\lambda^2 - 5 = 0.$$

By Cayley - Hamilton theorem,

$$A^2 - 5I = 0$$

Verification :

$$A^2 - 5I = 0$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 0$$

Hence proved.

(4) Use Cayley-Hamilton theorem to find the value of the matrix given by

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Soln :

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$$

$$C_1 = 2 + 1 + 2 = 5$$

$$C_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 7$$

$$C_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 3$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3 = 0 \rightarrow (1)$$

Given:

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3) + A(A^3 - 5A^2 + 8A - 2) + I$$

$$= A^5(0) + A[A^3 - 5A^2 + 7A - 3 + A + 1] + I$$

$$= 0 + A[0 + A + 1] + I$$

$$= A^2 + A + I$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

Problems:

(1) Reduce the quadratic form to a canonical form by an orthogonal reduction $2x_1x_2 + 2x_1x_3 - 2x_2x_3$. Also discuss its nature.

Soln:

$$Q = -2y_1^2 + y_2^2 + y_3^2$$

Index $= 2$

Rank $= 3$

Signature $= 1$

Nature $=$ Indefinite.