Method of residue:

$f(n)$ = Sum of the residues of $z^{n-1} F(z)$ at its poles

Note:

1. The pole of order 1 at $z=a$ is,

$$\left\{ \operatorname{Res}_{z=a} z^{n-1} F(z) \right\} = \lim_{z \to a} (z-a) \left[z^{n-1} F(z) \right]$$

2. The pole of order m at $z=a$ is,

$$\left\{ \operatorname{Res}_{z=a} z^{n-1} F(z) \right\} = \lim_{z \to a} \boxed{\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}} \left[z-a \right]^{m} z^{n-1} F(z)$$

Problems on cauchy's Residue method

1. Find $Z^{-1}\left[\dfrac{z}{(z-1)(z-2)}\right]$

Soln. :

$$\text{Let } F(z) = \frac{z}{(z-1)(z-2)}$$

$$z^{n-1} F(z) = z^{n-1} \frac{z}{(z-1)(z-2)} = \frac{z^{n}}{(z-1)(z-2)}$$

$n-1+1$

Here $z=1$ is a pole of order 1 and $z=2$ is a pole of order 1.

$Dr = 0$

$z-1=0$

$z-2=0$

$$\operatorname{Res}_{z=1}\left\{z^{n-1}F(z)\right\} = \lim_{z\to 1} (z-1)\frac{z^{n}}{(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{z^{n}}{z-2} = \frac{1}{1-2} = \frac{1}{-1}$$

$$= -1$$

$$\operatorname{Res}_{z=2}\left\{z^{n-1}F(z)\right\} = \lim_{z\to 2} (z-2)\frac{z^{n}}{(z-1)(z-2)}$$

$$= \lim_{z\to 2} \frac{z^{n}}{z-1} = \frac{2^n}{2-1}$$

$$= 2^{n}$$

$\therefore f(n) =$ Sum of the residues of $z^{n-1}F(z)$ at its pole

$$= -1 + 2^{n}$$

2. Find $z^{-1}\left[\dfrac{z(z+1)}{(z-1)^2}\right]$

Soln :

Let $F(z) = \dfrac{z(z+1)}{(z-1)^2}$

$$z^{n-1} F(z) = z^{n-1} \frac{z(z+1)}{(z-1)^2}$$

$$= \frac{z^n (z+1)}{(z-1)^2}$$

$z-1=0$

$z=1$

$m=2$

Here $z=1$ is a pole of order 2.

$$\operatorname{Res}_{z=1}\{z^{n-1}F(z)\} = \lim_{z\to 1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[(z-1)^2 \frac{z^n (z+1)}{(z-1)^2}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[z^n (z+1)\right]$$

$$= \lim_{z\to 1} \left[z^n (1) + (z+1)\, n z^{n-1}\right]$$

$n! = n(n-1).$

$5! = 5\times 4\times 3\times 2\times$

$$= \lim_{z\to 1} \left[1^n + n(2)(1)^{n-1}\right]$$

$$= 1+2n$$

$\therefore f(n) =$ Sum of the residues

$= 1+2n$

3]. Find $Z^{-1}\left[\dfrac{z^2}{z^2+4}\right]$ using Cauchy's Residue method.

Soln. :

Let $F(z) = \dfrac{z^2}{z^2+4}$

$$z^{n-1} F(z) = z^{n-1} \frac{z^2}{z^2+4}$$

$$= \frac{z^{n+1}}{z^2+4} \qquad \left[\begin{array}{l} z^2+4=0 \\ z^2=-4 \\ z=\pm 2i \end{array} \right.$$

Here $z = 2i$ is a pole of order 1.

and $z = -2i$ is a pole of order 1.

$$\operatorname{Res}[z^{n-1} F(z)] = \lim_{z \to 2i} (z-2i) \frac{z^{n+1}}{z^2+4}$$

$$= \lim_{z \to 2i} (z-2i) \frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$= \frac{(2i)^{n+1}}{2i+2i} = \frac{(2i)^n (2i)}{4i}$$

$$= \frac{(2i)^n}{2} = \frac{2^n i^n}{2}$$

$$= 2^{n-1} i^n$$

$$\operatorname{Res}[z^{n-1} F(z)] = \lim_{z \to -2i} (z-(-2i)) \frac{z^{n+1}}{z^2+4}$$

$$= \lim_{z \to -2i} (z+2i) \frac{z^{n+1}}{(z-2i)(z+2i)}$$

$$= \frac{(-2i)^{n+1}}{-2i-2i}$$

$$= \frac{(-2i)^n (-2i)}{-4i}$$

$$= \frac{(-2i)^n}{2} = \frac{2^n (-i)^n}{2}$$

$$= 2^{n-1} (-i)^n$$

$\therefore f(n) =$ Sum of the residues

$$= 2^{n-1}(i)^n + 2^{n-1}(-i)^n$$

$$= 2^{n-1}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

$$= 2^{n-1}\left[2\cos\frac{n\pi}{2}\right]$$

$$= 2^n \cos\frac{n\pi}{2}$$

HW 1. $Z^{-1}\left[\frac{z}{(z-1)(z^2+1)}\right]$