Z - TRANSFORM & DIFFERENTIAL EQUATION

Defn: z - transform [Two sided (or) bilateral]

Let $\{f(n)\}$ be a sequence defined for all integers then its z - transform is defined to be

$$F(z) = Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}.$$

where z is an arbitrary complex number.

Defn: z - transform [One - sided (or) unilateral]

Let $\{f(n)\}$ be a sequence defined for all positive integers then the z - transform of $\{f(n)\}$ is defined to be

$$F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

Defn: z - transform for discrete values of t.

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, \ldots$ T being the sampling period, then z - transform of $f(t)$ is defined as $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$.

Note:

1) If $f(n)$ given then replace 'n' by 'n'
2) If $f(t)$ given then replace 't' by 'nT'.

If $z[f(n)] = F(z)$ then $z^{-1}[F(z)] = f(n)$

1) $z[a^n] = \dfrac{z}{z-a} \Rightarrow a^n = z^{-1}\left[\dfrac{z}{z-a}\right]$

2) $z[(-a)^n] = \dfrac{z}{z+a} \Rightarrow (-a)^n = z^{-1}\left[\dfrac{z}{z+a}\right]$

3) $z[n] = \dfrac{z}{(z-1)^2} \Rightarrow (n) = z^{-1}.$

4) $z[na^n] = \dfrac{az}{(z-a)^2}$

5) $z[na^{n-1}] = \dfrac{z}{(z-a)^2}$

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6) $z[a^{n-1}] = \frac{1}{z-a}$

7) $z[(-a)^{n-1}] = \frac{1}{z+a}$

8) $z[n(-a)^{n-1}] = \frac{z}{(z+a)^2}$

9) $z[(n-1)a^{n-2}] = \frac{1}{(z-a)^2}$

10) $z[(n-1)(-a)^{n-2}] = \frac{1}{(z+a)^2}$

~~11)~~ $\cancel{z[e^{1/z}] = \frac{1}{n!}}$

11) $z\left[\frac{1}{n+1}\right] = -z\log(1-1/z)$

12) $z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$

13) $z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

14) $z\left[\frac{1}{n!}\right] = e^{1/z}$

15) $z[n^2] = \frac{z(z+1)}{(z-1)^3}$

1) Find the z-transform of 1 or $z(1)$.

(or) P.T. $z(1) = \frac{z}{z-1}$, $|z| > 1$.

We know $z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$z(1) = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \quad \left[(1-x)^{-1} = 1 + x + x^2 + \cdots\right]$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$z(1) = \frac{z}{z-1}$$

② Find $z(-1)^n$.

$$z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$z(-1)^n = \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n$$

$$= 1 - \frac{1}{z} + \left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3 + \cdots$$

$$= 1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots$$

$$= \left(1 + \frac{1}{z}\right)^{-1} \quad \left[(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right]$$

$$z(-1)^n = \frac{z}{z+1}$$

PROPERTIES OF Z-TRANSFORM

(i) Linear property:

$$z[af(n) + bg(n)] = aF(z) + bG(z)$$

$$z[af(n)+bg(n)] = \sum_{n=0}^{\infty} (af(n)+bg(n))z^{-n}.$$

$$= \sum_{n=0}^{\infty} a f(n) z^{-n} + \sum_{n=0}^{\infty} b g(n) z^{-n}$$

$$= aF(z) + bG(z)$$

(ii) First shifting Theorem:

If $z[f(t)] = F(z)$ then $z[e^{-at} f(t)] = F[ze^{aT}]$

$$z[e^{-at} f(t)] = F\left[e^{-anT} f(nT)\right]$$

$$= \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}.$$

$$= \sum_{n=0}^{\infty} f(nT)(ze^{aT})^{-n}$$

$$= F(ze^{aT}).$$

(v) Differentiation in z domain :

$$Z[n f(n)] = -z \frac{d}{dz}[\{F(z)\}]$$

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\frac{d}{dz} F(z) = \frac{d}{dz} \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) z^{-n-1} (-n)$$

$$= \sum_{n=0}^{\infty} f(n) z^{-(n)} \cdot z^{-1} (-n)$$

$$= -z^{-1} \sum_{n=0}^{\infty} f(n)\, n\, z^{-n}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n f(n) z^{-n}$$

$$= -\frac{1}{z}\left[\sum_{n=0}^{\infty} n f(n) z^{-n}\right]$$

$$= -\frac{1}{z}\left[Z[n f(n)]\right]$$

$$\Rightarrow \frac{d}{dz} F(z) = -\frac{1}{z}\left[Z[n f(n)]\right]$$

$$\Rightarrow Z[n f(n)] = -z \frac{d}{dz}[F(z)]$$

1) Find $z[e^{-iat}]$

$$z[e^{-iat}] = z[1 \cdot e^{-iat}]$$

$$= [z(1)]_{z \to z e^{+iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to z e^{iaT}}$$

$$= \frac{z e^{iaT}}{z e^{iaT} - 1}$$

2) Find Z-Transform of $\cos at$ & $\sin at$

$z[\cos at]$ & $z[\sin at]$

$$z[e^{iat}] = z[1 \cdot e^{iat}]$$

$$= [z(1)]_{z \to z e^{-iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to z e^{-iaT}}$$

$$= \frac{z e^{-iaT}}{z e^{-iaT} - 1}$$

Divide by e^{-iaT}

$$= \frac{z}{z\cancel{e^{-iaT}} - e^{iaT}}$$

$$= \frac{z}{z - [\cos aT + i \sin aT]}$$

$$= \frac{z}{(z - \cos aT) - i \sin aT}$$

$$z[\cos at + i \sin at] = \frac{z[z - \cos aT + i \sin aT]}{(z - \cos aT)^2 + (\sin aT)^2}$$ [Taking conjugate]

$$\Rightarrow z[\cos at] = \frac{z[z - \cos aT]}{(z - \cos aT)^2 + (\sin aT)^2} = \frac{z[z - \cos aT]}{z^2 - 2z\cos aT + 1}$$

$$\Rightarrow z[\sin at] = \frac{z \sin aT}{z^2 - 2z\cos aT + 1}$$

2) Find the initial & final value of $F(z) = \frac{z}{2z^2-3z+1}$

Initial value: $\lim_{z\to\infty} F(z) = \lim_{t\to 0} f(t)$

$$\lim_{z\to\infty}\left[\frac{z}{2z^2-3z+1}\right] = f(0)$$

$$\lim_{z\to\infty}\frac{1/z^2}{1/z^2}\left[\frac{z}{2z^2-3z+1}\right] = f(0)$$

$$\lim_{z\to\infty}\left[\frac{1/z}{2-3/z+1/z^2}\right] = f(0)$$

$$0 = f(0)$$

Final value: $\lim_{z\to 1}(z-1)F(z) = f(\infty)$

$$\lim_{z\to 1}(z-1)\left(\frac{z}{2z^2-3z+1}\right) = f(\infty)$$

$$\lim_{z\to 1}\cancel{(z-1)}\frac{z}{\cancel{(z-1)}(2z-1)} = f(\infty)$$

$$1 = f(\infty).$$

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Method of Partial Fraction:

1) Find inverse z-Transform of $\frac{z-4}{(z+2)(z+3)}$ using partial fraction method (or)

Find $z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right]$.

$$\frac{z-4}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$$

$$z-4 = A(z+3) + B(z+2)$$

put $z=-3 \Rightarrow B=7$

put $z=-2 \Rightarrow A=-6$

$$\frac{z-4}{(z+2)(z+3)} = -\frac{6}{z+2} + \frac{7}{z+3}$$

$$z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right] = z^{-1}(-6)\left[\frac{1}{z+2}\right] + 7z^{-1}\left[\frac{1}{z+3}\right].$$

$$= -6(-2)^{n-1} + 7(-3)^{n-1}.$$

2) Find $z^{-1}\left[\frac{4z^2-2z}{(z-1)(z-2)^2}\right]$

$$\frac{4z^2-2z}{(z-1)(z-2)^2} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}.$$

$$4z^2-2z = A(z-2)^2 + B(z-2)(z-1) + C(z-1).$$

put $z=2 \Rightarrow C=12$

put $z=1 \Rightarrow A=2$

put $z=0 \Rightarrow B=2$.

$$\frac{4z^2-2z}{(z-1)(z-2)^2} = \frac{2}{(z-1)} + \frac{2}{(z-2)} + \frac{12}{(z-2)^2}$$

$$z^{-1}\left[\frac{4z^2-2z}{(z-1)(z-2)^2}\right] = 2z^{-1}\left[\frac{1}{z-1}\right] + 2z^{-1}\left[\frac{1}{z-2}\right] + 12z^{-1}\left[\frac{1}{(z-2)^2}\right]$$

$$= 2(1)^{n-1} + 2(2)^{n-1} + 12(n-1)(2)^{n-2}.$$

3) Find $z^{-1}\left[\frac{z^2-3z}{(z+5)(z-2)}\right]$

$$F(z) = \frac{z^2-3z}{(z+5)(z-2)}$$

$$\frac{F(z)}{z} = \frac{z-3}{(z+5)(z-2)}$$

$$\frac{z-3}{(z+5)(z+2)} = \frac{A}{(z+5)} + \frac{B}{(z-2)}$$

$$z-3 = A(z-2)+B(z+5)$$

put $z=2 \Rightarrow B = -1/7$

put $z=-5 \Rightarrow A = 8/7$.

$$\frac{F(z)}{z} = \frac{8/7}{(z+5)} - \frac{1/7}{(z-2)}$$

$$\Rightarrow F(z) = \frac{8}{7}\left(\frac{z}{z+5}\right) - \frac{1}{7}\cdot\left(\frac{z}{z-2}\right).$$

$$z^{-1}\left[\frac{z^2-3z}{(z+5)(z-2)}\right] = \frac{8}{7}z^{-1}\left[\frac{z}{z+5}\right] - \frac{1}{7}z^{-1}\left[\frac{z}{z-2}\right]$$

$$= \frac{8}{7}(-5)^n - \frac{1}{7}(2)^n.$$

Another method:

$$\frac{z^2-3z}{(z+5)(z-2)} = \frac{A}{(z+5)} + \frac{B}{(z-2)}.$$

$$z^2-3z = A(z-2)+B(z+5)$$

put $z=2 \Rightarrow B = -2/7$

put $z=-5 \Rightarrow A = -40/7$.

$$\frac{z^2-3z}{(z+5)(z-2)} = -\frac{40/7}{(z+5)} - \frac{2/7}{(z-2)}$$

$$z^{-1}\left[\frac{z^2-3z}{(z+5)(z-2)}\right] = -\frac{40}{7}z^{-1}\left(\frac{1}{z+5}\right) - \frac{2}{7}z^{-1}\left(\frac{1}{z-2}\right)$$

$$= -\frac{40}{7}(-5)^{n-1} - \frac{2}{7}(2)^{n-1}$$

$$z^{-1}\left[\frac{z^2}{(z^2+1)(z+3)}\right] = z^{-1}\,\frac{3}{10}\left(\frac{z^2}{z^2+1}\right) + \frac{1}{10}\,z^{-1}\left(\frac{z}{z^2+1}\right) - \frac{3}{10}\,z^{-1}\left(\frac{z}{z+3}\right)$$

$$= \frac{3}{10}\cos\frac{n\pi}{2} + \frac{1}{10}\sin\frac{n\pi}{2} - \frac{3}{10}(-3)^n.$$

H.W. Find z^{-1} of

(1) $\dfrac{z}{(z-1)(z-2)^2}$ $\left((1)^{n-1} - (2)^{n-1} + 2(n-1)2^{n-2}\right)$

(2) $\dfrac{z^3}{(z-1)^2(z-2)}$ $\left(-3(1)^n - n(1)^{n-1} + 4(2)^n\right)$

(3) $\dfrac{z^2-z+2}{(z+1)(z-1)^2}$ $\left((-1)^{n-1} + (n-2)(1)^{n-1}\right)$

(4) $\dfrac{z^2-z+2}{(z^2-1)(z-1)} = \dfrac{z^2-z+2}{(z+1)(z-1)^2}$.

$(z+1)(z-1)(z-1)$

(5) $\dfrac{z^2+3z}{(z-1)(z-2)(z-3)}$ $\left(-2(1)^{n-1} - 10(2)^{n-1} + 9(3)^{n-1}\right)$

(6) $\dfrac{z^2}{(z+2)(z^2+4)}$ $\left(-\frac{1}{4}(-2)^n + \frac{1}{4}2^n\cos\frac{n\pi}{2} + \frac{1}{4}2^n\sin\frac{n\pi}{2}\right)$

(7) Find $z^{-1}\left[\dfrac{4z^2-z}{z^3-5z^2+8z-4}\right]$

$$F(z) = \frac{4z^2-z}{z^3-5z^2+8z-4}$$

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\Rightarrow (x+1)(x^2-4x+4)=0$

$\Rightarrow (x+1)(x-2)^2$.

$$\Rightarrow \frac{F(z)}{z} = \frac{\cancel{z}[4z-1]}{(z+1)(z-2)^2}$$

$$\Rightarrow \frac{4z-1}{(z+1)(z-2)^2} = \frac{A}{(z+1)} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

$$\Rightarrow 4z-1 = A(z-2)^2 + B(z-2)(z+1) + C(z+1)$$

put $z=-1 \Rightarrow A = -5/9$

$z = 2 \Rightarrow C = 7/3$

$z = 0 \Rightarrow B = 5/9$.

$$\frac{F(z)}{z} = \frac{-5/9}{(z+1)} + \frac{5/9}{(z-2)} + \frac{7/3}{(z-2)^2}.$$

$$F(z) = \frac{4z^2-z}{z^3-5z^2+8z-4} = -\frac{5}{9}\left(\frac{z}{z+1}\right) + \frac{5}{9}\left(\frac{z}{z-2}\right) + \frac{7}{3}\left(\frac{z}{(z-2)^2}\right)$$

$$z^{-1}\left[\frac{4z^2-z}{z^3-5z^2+8z-4}\right] = -\frac{5}{9}z^{-1}\left(\frac{z}{z+1}\right) + \frac{5}{9}z^{-1}\left(\frac{z}{z-2}\right) + \frac{7}{3}z^{-1}\left(\frac{z}{(z-2)^2}\right)$$

$$= -\frac{5}{9}(-1)^n + \frac{5}{9}(2)^n + \frac{7}{3}n(2)^{n-1}.$$

4) Find $z^{-1}\left[\frac{z^3}{(z-2)^2(z+2)}\right]$

$$F(z) = \frac{z^3}{(z-2)^2(z+2)}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-2)^2(z+2)}$$

$$\Rightarrow \frac{z^2}{(z-2)^2(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z-2)^2} + \frac{C}{(z+2)}$$

$$\Rightarrow z^2 = A(z-2)(z+2) + B(z+2) + C(z-2)^2$$

put $z = 2 \Rightarrow B = 1$

put $z = -2 \Rightarrow C = 1/4$

put $z = 0 \Rightarrow A = 3/4$

$$F(z) = \frac{z^3}{(z-2)^2(z+2)} = \frac{3/4\,z}{(z-2)} + \frac{1\cdot z}{(z-2)^2} + \frac{1/4\,z}{(z+2)}$$

$$z^{-1}\left[\frac{z^3}{(z-2)^2(z+2)}\right] = \frac{3}{4} z^{-1}\left[\frac{z}{z-2}\right] + z^{-1}\left[\frac{z}{(z-2)^2}\right] + \frac{1}{4} z^{-1}\left[\frac{z}{z+2}\right].$$

$$= 3/4\,(2)^n + n\,(2)^{n-1} + 1/4\,(-2)^n.$$

5) Find $z^{-1}\left[\frac{z^2}{(z^2+1)(z+3)}\right]$

$$\frac{F(z)}{z} = \frac{z}{(z^2+1)(z+3)}$$

$$\frac{z}{(z^2+1)(z+3)} = \frac{Az+B}{(z^2+1)} + \frac{C}{z+3}.$$

$$z = (Az+B)(z+3) + C(z^2+1)$$

put $z = -3 \Rightarrow C = -3/10$.

put $z = 0 \Rightarrow B = 1/10$.

put $z = 1 \Rightarrow A = 3/10$.

$$\frac{F(z)}{z} = \frac{3/10\,z + 1/10}{z^2+1} - \frac{3/10}{z+3}.$$