Unit - V

Z-transforms and Difference Equations

Z-transform - Elementary Properties - Inverse Z-transform - Convolution Theorem - Formation of difference equation - Solution of difference equations using Z-transform.

Z-transforms and Difference Equations

Z-transform:

Let $\{f(n)\}$ be a sequence defined for $n = 0, 1, 2, \ldots$ and $f(n) = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z[f(n)] = F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

Note:

$$Z[f(t)] = F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

Inverse Z-transform:

If $Z[f(n)] = F(z)$, then $Z^{-1}[F(z)] = f(n)$ is called inverse Z-transform of $F(z)$.

Formulas on Z-transform:	Inverse Z-transform
1. $Z[1] = \dfrac{z}{z-1}$	$Z^{-1}\left[\dfrac{z}{z-1}\right] = 1$
2. $Z[a^n] = \dfrac{z}{z-a}$	$Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$
3. $Z[a^{n-1}] = \dfrac{1}{z-a}$	$Z^{-1}\left[\dfrac{1}{z-a}\right] = a^{n-1}$
4. $Z[(n-1)a^{n-1}] = \dfrac{a}{(z-a)^2}$	$Z^{-1}\left[\dfrac{a}{(z-a)^2}\right] = (n-1)\,a^{n-1}$

5. $Z[n] = \frac{z}{(z-1)^2}$ | $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

6. $Z[n^2] = \frac{z(z+1)}{(z-1)^3}$ | $Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = n^2$

7. $Z[na^n] = \frac{az}{(z-a)^2}$ | $Z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$

8. $Z[na^{n-1}] = \frac{z}{(z-a)^2}$ | $Z^{-1}\left[\frac{z}{(z-a)^2}\right] = na^{n-1}$

9. $Z\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$ | $Z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos\frac{n\pi}{2}$

10. $Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$ | $Z^{-1}\left[\frac{z}{z^2+1}\right] = \sin\frac{n\pi}{2}$

11. $Z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}$ | $Z^{-1}\left[z\log\frac{z}{z-1}\right] = \frac{1}{n+1}$

12. $Z\left[\frac{1}{n-1}\right] = \frac{1}{z}\log\frac{z}{z-1}$ | $Z^{-1}\left[\frac{1}{z}\log\frac{z}{z-1}\right] = \frac{1}{n-1}$

13. $Z\left[\frac{1}{(n+1)!}\right] = z\left[e^{\frac{1}{z}}-1\right]$ | $Z^{-1}\left[z(e^{1/z}-1)\right] = \frac{1}{(n+1)!}$

1. Prove that $Z[1] = \frac{z}{z-1}$, $|z|>1$

Proof:

WKT $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z[1] = \sum_{n=0}^{\infty} 1\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left(1-\frac{1}{z}\right)^{-1} = \left(\frac{z-1}{z}\right)^{-1}$$

$$\therefore \quad Z[1] = \frac{z}{z-1}$$

Inverse Z-transform

Methods :

1. Partial fraction method
2. Cauchy's Residue method

Partial fraction Method :

Type I :

$$\frac{1}{(z+a)(z+b)} = \frac{A}{(z+a)} + \frac{B}{(z+b)}$$

Type II :

$$\frac{1}{(z+a)^2 (z+b)} = \frac{A}{z+a} + \frac{B}{(z+a)^2} + \frac{C}{z+b}$$

Type III :

$$\frac{1}{(z^2+a)(z+b)} = \frac{Az+B}{z^2+a} + \frac{C}{z+b}$$

Problems :

1. find $z^{-1}\left[\frac{10z}{z^2-3z+2}\right]$

Soln. :

Let $F(z) = \frac{10z}{z^2-3z+2}$

$$\frac{F(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (1)$$

$$\frac{10}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$10 = A(z-2) + B(z-1)$$

when $z=1$, $A=-10$

$z=2$, $B=10$

$$\therefore \frac{F(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$(1) \Rightarrow \frac{F(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2}$$

$$F(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$z^{-1}[F(z)] = z^{-1}\left[\frac{-10z}{z-1}\right] + z^{-1}\left[\frac{10z}{z-2}\right]$$

$$= -10\, z^{-1}\left[\frac{z}{z-1}\right] + 10\, z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -10(1)^n + 10(2)^n$$

2]. Find $z^{-1}\left[\frac{z^2-3z}{(z-5)(z+2)}\right]$

Soln. :

Let $F(z) = \dfrac{z^2-3z}{(z-5)(z+2)}$

$$\frac{F(z)}{z} = \frac{z-3}{(z-5)(z+2)}$$

$$\frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2} \longrightarrow (1)$$

$$= \frac{A(z+2)+B(z-5)}{(z-5)(z+2)}$$

$$z-3 = A(z+2)+B(z-5)$$

when $z=-2$, $\quad -5 = -7B$

$$B = 5/7$$

when $z=5$, $\quad 2 = 7A \Rightarrow A = 2/7$

$$\frac{F(z)}{z} = \frac{2/7}{z-5} + \frac{5/7}{z+2}$$

$$F(z) = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

$$\therefore \; Z^{-1}[F(z)] = \frac{2}{7} Z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7} Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n$$

3]. Find $Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$

Soln.

Let $F(z) = \frac{z^3}{(z-1)^2(z-2)}$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$\frac{z^2}{(z-1)^2(z-2)} = \frac{A(z-1)(z-2)+B(z-2)+C(z-1)^2}{(z-1)^2(z-2)}$$

$$z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

when $z = 1$, $\quad B = -1$

$z = 2$, $\quad C = 4$

$z = 0$, $\quad 0 = 2A - 2B + C$

$A = -3$

$$(1) \Rightarrow \quad \frac{F(z)}{z} = \frac{-3}{z-1} - \frac{1}{(z-1)^2} + \frac{4}{z-2}$$

$$\frac{z^{\cancel{2}\,3}}{(z-1)^2(z-2)} = -3\,\frac{z}{z-1} - \frac{z}{(z-1)^2} + 4\,\frac{z}{z-2}$$

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = -3\,Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4\,Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -3(1) - n + 4(2)^n$$

$$= 3 - n + 4.2^n.$$

4]. Find $Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$

Soln. :

Let $F(z) = \frac{z^2}{(z+2)(z^2+4)}$

$$\frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)} \rightarrow (1)$$

Now $\frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4} \rightarrow (2)$

$$z = A(z^2+4) + (Bz+C)(z+2)$$

when $z=-2$, $\quad -2 = 8A + (-2B+C)(0)$

$$A = -1/4$$

$z=0$, $\quad 0 = 4A + 2C$

$$C = 1/2$$

Equate the coefficient of z^2,

$$A+B=0$$

$$B = 1/4$$

$\therefore$ (2) $\Rightarrow$ $\frac{z}{(z+2)(z^2+4)} = \frac{-1/4}{z+2} + \frac{1/4 z + 1/2}{z^2+4}$

$$(1) \Rightarrow \quad F(z) = \frac{-1}{4} \, \frac{z}{z+2} + \frac{1}{4} \, \frac{z^2}{z^2+4} + \frac{1}{2} \, \frac{z}{z^2+4}$$

$$Z^{-1}[F(z)] = \frac{-1}{4} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4} Z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{2(2)} Z^{-1}\left[\frac{2z}{z^2+4}\right]$$

$$= \frac{-1}{4} (-2)^n + \frac{1}{4} 2^n \cos \frac{n\pi}{2} + \frac{1}{4} 2^n \sin \frac{n\pi}{2}$$

$$\because \quad Z\left[a^n \cos \frac{n\pi}{2}\right] = \frac{z^2}{z^2+a^2}$$

$$Z\left[a^n \sin \frac{n\pi}{2}\right] = \frac{az}{z^2+a^2}$$