Convolution of Two Sequences:

The convolution of two functions $f(n)$ and $g(n)$ is defined as,

$$f(n) * g(n) = \sum_{r=0}^{n} f(r)\, g(n-r)$$

Convolution Theorem:

If $Z[f(n)] = F(z)$ and $Z[g(n)] = G(z)$,

then $$Z[f(n) * g(n)] = F(z) \cdot G(z)$$

$$\Rightarrow Z^{-1}[F(z) \cdot G(z)] = f(n) * g(n)$$

Note:

I. $\sum_{r=0}^{n} a^n = a^n (n+1)$

2. $1 + a + a^2 + \dots + a^n = \dfrac{a^{n+1} - 1}{a - 1}$ if $a > 1$

$= \dfrac{1 - a^{n+1}}{1 - a}$ if $a < 1$.

Problems on Convolution:

1]. Find the Z-transform of $f(n) * g(n)$ where $f(n) = \left(\frac{1}{2}\right)^n$ and $g(n) = \cos n\pi$ using convolution theorem.

Soln.:

By convolution theorem,

$$Z[f(n) * g(n)] = Z[f(n)] \cdot Z[g(n)] \rightarrow (1)$$

Now, $Z[f(n)] = Z\left[\left(\frac{1}{2}\right)^n\right]$

$$= \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}$$

and $Z[g(n)] = Z[\cos n\pi]$

$$= Z[(-1)^n]$$

$$= \frac{z}{z-(-1)}$$

$$= \frac{z}{z+1}$$

$$(1) \Rightarrow Z[f(n) * g(n)] = \frac{2z}{2z-1}\left(\frac{z}{z+1}\right)$$

$$= \frac{2z^2}{(2z-1)(z+1)}$$

2]. Use convolution, find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

Soln.:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$ By convolution theorem

$$= a^n * a^n$$

$$= \sum_{r=0}^{n} a^r \, a^{n-r}$$ [By defn. of convolution]

$$= \sum_{r=0}^{n} a^n$$

$$= (n+1)\,a^n$$

2]. Using convolution theorem, find

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right].$$

Soln. :

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n * 3^n$$

$$= \sum_{r=0}^{n} (1)^r \, 3^{n-r}$$

$$= \sum_{r=0}^{n} 3^{n-r}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + \dots + 3^0$$

$$= 1 + 3 + \dots + 3^{n-2} + 3^{n-1} + 3^n$$

$$= \frac{3^{n+1}-1}{3-1}$$

$$= \frac{3^{n+1}-1}{2}$$

Scanned with CamScanner

3]. Using convolution theorem, find

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right].$$

Soln:

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-4}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 4^n * 3^n$$

$$= \sum_{r=0}^{n} 4^r\, 3^{n-r}$$

$$= 3^n \sum_{r=0}^{n} 4^r\, 3^{-r} = 3^n \sum_{r=0}^{n} \left(\frac{4}{3}\right)^r$$

$$= 3^n\left[1 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^n\right]$$

$$= 3^n\left[\frac{(4/3)^{n+1} - 1}{4/3 - 1}\right]$$

$$= 3^n\,\frac{3}{1}\left[(4/3)^{n+1} - 1\right]$$

$$= 3^{n+1}\left[\left(\frac{4}{3}\right)^{n+1} - 1\right] = 3^{n+1}\left[\frac{4^{n+1} - 3^{n+1}}{3^{n+1}}\right]$$

$$= 4^{n+1} - 3^{n+1}$$

4]. Use convolution theorem, find

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$$

Soln. :

$$Z^{-1}\left[\frac{8z^2}{2(z-\frac{1}{2})4(z+\frac{1}{4})}\right] = Z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z+\frac{1}{4})}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}} \cdot \frac{z}{z+\frac{1}{4}}\right]$$

$$= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z+\frac{1}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(-\frac{1}{4}\right)^n$$

$$= \sum_{r=0}^{n} \left(\frac{1}{2}\right)^r \left(-\frac{1}{4}\right)^{n-r}$$

$$= \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n} \left(\frac{1}{2}\right)^r \left(-\frac{1}{4}\right)^{-r}$$

$$= \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n} \left(\frac{1}{2}\right)^r (-4)^r$$

$$= \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n} \left[-4\left(\frac{1}{2}\right)\right]^r = \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n} (-2)^r$$

$$= \left(-\frac{1}{4}\right)^n \left[1 + (-2) + (-2)^2 + \dots + (-2)^n\right]$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{1-(-2)}\right]$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{3}\right]$$

$$= \left(-\frac{1}{4}\right)^n \left[\frac{1+2(-2)^n}{3}\right]$$

5]. using convolution, find

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

Soln. :

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{r=0}^{n} a^r \, b^{n-r}$$

$$= b^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r$$

$$= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right]$$

$$= b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$

$$= b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b}\right]$$

$$= b^n \left[\frac{a^{n+1} - b^{n+1}}{b^n (a-b)}\right]$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

Hw 1]. $Z^{-1}\left[\frac{14z^2}{(7z-1)(2z-1)}\right]$ 2]. $Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$