Two-dimensional heat equation

The differential equation for two dimensional heat flow for the unsteady case is

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state u is independent of 't' (ie) $\frac{\partial u}{\partial t} = 0$.

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad [\text{Laplace Equation}]$$

Possible solutions:

i) $u(x,y) = (A_1 e^{\lambda x} + A_2 e^{-\lambda x})(A_3 \cos\lambda y + A_4 \sin\lambda y)$

ii) $u(x,y) = (A_5 \cos\lambda x + A_6 \sin\lambda x)(A_7 e^{\lambda y} + A_8 e^{-\lambda y})$

iii) $u(x,y) = (A_9 x + A_{10})(A_{11} y + A_{12})$

Suitable solution:

i) If heat flows in x-direction, then

$$u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$$

ii) If heat flows in y-direction, then

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$$

Types of plates:

1) Finite plates

i) Square plate

ii) Rectangular plate

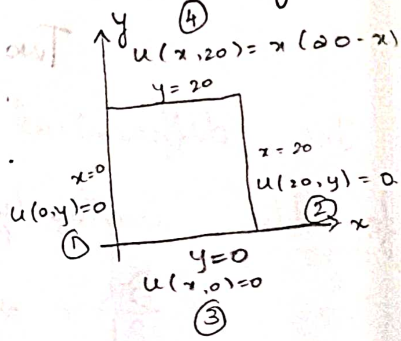
2) Infinite plates

i) Vertically infinite plate

ii) Horizontally infinite plate

Finite plates:

1) The square plate bounded by the line $x=0, y=0, x=20, y=20$. It's faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20) = x(20-x)$ while the other three edges are kept at $0°c$. Find the steady state temperature in the plate.



The Laplace eq is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

The boundary conditions are,

i) $u(0,y)=0$

ii) $u(20,y)=0$

iii) $u(x,0)=0$

iv) $u(x,20) = x(20-x)$

The suitable soln is,

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \quad -①$$

Apply (i) in ①

$$u(0,y) = (A\cos\lambda(0) + B\sin\lambda(0))(Ce^{\lambda y} + De^{-\lambda y})$$

$$0 = A(Ce^{\lambda y} + De^{-\lambda y})$$

$Ce^{\lambda y} + De^{-\lambda y} \neq 0$ $\Rightarrow A=0$. sub in ①.

$$u(x,y) = B\sin\lambda x\,(Ce^{\lambda y} + De^{-\lambda y}) \quad -②$$

Apply (ii) in ②.

$$u(20,y) = B\sin\lambda(20)(Ce^{\lambda y} + De^{-\lambda y})$$

$$0 = B\sin\lambda(20)(Ce^{\lambda y} + De^{-\lambda y})$$

$B \neq 0$, $Ce^{\lambda y} + De^{-\lambda y} \neq 0$ $\Rightarrow \sin\lambda(20) = 0$

$$20\lambda = n\pi$$

$$\lambda = \frac{n\pi}{20}$$

sub in ②.

$$\therefore u(x,y) = B\sin\frac{n\pi x}{20}\left[Ce^{\frac{n\pi y}{20}} + De^{\frac{-n\pi y}{20}}\right] \quad -③$$

Apply (iii) in ③.

$$u(x,0) = B\sin\frac{n\pi x}{20}\left[Ce^{(0)} + De^{-(0)}\right]$$

$$0 = B\sin\frac{n\pi x}{20}[C+D]$$

$B \neq 0$, $\sin\frac{n\pi x}{20} \neq 0$ $\Rightarrow C+D=0$ $\Rightarrow D=-C$.

sub in ③. $\Rightarrow u(x,y) = B\sin\frac{n\pi x}{20}\left[Ce^{\frac{n\pi y}{20}} - Ce^{-\frac{n\pi y}{20}}\right]$

$$u(x,y) = BC\sin\frac{n\pi x}{20}\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20} \qquad \left[\begin{array}{l}\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \\ \Rightarrow 2\sinh\theta = e^{\theta}-e^{-\theta}\end{array}\right]$$

where $2BC = B_n$.

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20} \quad -④$$

Apply (iv) in ④.

$$u(x,20) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{20}\sinh n\pi$$

$$x(20-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} \quad \left[\because B_n\sinh n\pi = b_n\right]$$

Now, Applying Half Range Sine Series,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{20}\int_0^{20}(20x-x^2)\sin\frac{n\pi x}{20}\,dx$$

$$= \frac{1}{10}\int_0^{20}(20x-x^2)\sin\frac{n\pi x}{20}\,dx.$$

$u = 20x - x^2$	$v = \sin\frac{n\pi x}{20}$
$u' = 20-2x$	$v_1 = \frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}$
$u'' = -2.$	$v_2 = \frac{-\sin\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2}$
$u''' = 0$	$v_3 = \frac{\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^3}$

$$= \frac{1}{10}\left[(20x - x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) + (20-2x)\frac{\sin\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2} - 2\frac{\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^3}\right]_0^{20}$$

$$= \frac{1}{10}\left[0 + 0 - 2\frac{\cos n\pi}{\frac{n^3\pi^3}{8000}} + 2\frac{(1)}{\frac{n^3\pi^3}{8000}}\right]$$

$$= \frac{1}{10}\frac{8000}{n^3\pi^3}2\left[1-(-1)^n\right]$$

$$b_n = \frac{1600}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n \sinh n\pi = \frac{1600}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{1600}{n^3\pi^3\sinh n\pi}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} \frac{3200}{n^3\pi^3\sinh n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

sub in (4) $\Rightarrow u(x,y) = \sum_{n=odd}^{\infty} \frac{3200}{n^3\pi^3\sinh n\pi} \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$

$$u(x,y) = \frac{3200}{\pi^3}\sum_{n=odd}^{\infty}\frac{1}{n^3\sinh n\pi}\sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20}$$