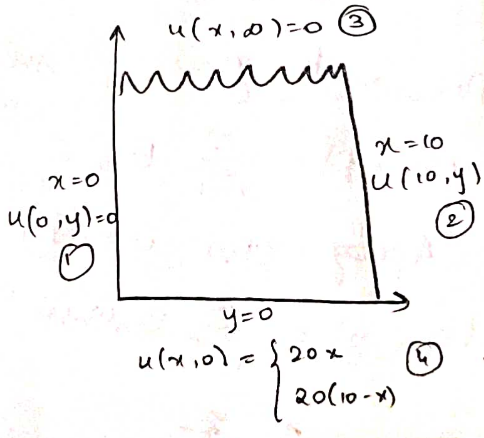
2. A rectangular plate with insulated surface is 10cm wide and so long compared to its width that it may be considered as infinite without introducing the error. The temperature at short edge $y=0$ is given by,

$$u = \begin{cases} 20x & , 0 \le x \le 5 \\ 20(10-x) & , 5 \le x \le 10 \end{cases}$$

and the two long edges $x=0$ and $x=10$ as well as the other short edges are kept at $0°c$. Find $u(x,y)$

The Laplace eq is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.



The boundary conditions are,

i) $u(0,y)=0$

ii) $u(10,y)=0$

iii) $u(x,\infty)=0$

iv) $u(x,0)= \begin{cases} 20x & , 0 \le x \le 5 \\ 20(10-x) & , 5 \le x \le 10 \end{cases}$

The suitable solution is

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(ce^{\lambda y} + De^{-\lambda y}) \quad -(1)$$

Apply (i) in (1)

$$u(0,y) = (A(1)+B(0))(ce^{\lambda y}+De^{-\lambda y})$$

$$0 = A(ce^{\lambda y}+De^{-\lambda y})$$

$ce^{\lambda y}+De^{-\lambda y} \neq 0 \quad \Rightarrow \boxed{A=0}$

sub in (1).

$$\therefore u(x,y) = B\sin\lambda x\,(ce^{\lambda y}+De^{-\lambda y}) \quad -(2)$$

Apply (ii) in (2)

$$u(10,y) = B\sin\lambda(10)\,(ce^{\lambda y}+De^{-\lambda y})$$

$$0 = B\sin 10\lambda\,(ce^{\lambda y}+De^{-\lambda y})$$

$ce^{\lambda y}+De^{-\lambda y} \neq 0, \quad B \neq 0 \quad . \Rightarrow \sin 10\lambda = 0$

$10\lambda = n\pi$

$\boxed{\lambda = \frac{n\pi}{10}}$

sub in (2).

$$\therefore u(x,y) = B\sin\frac{n\pi x}{10}\left(ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) \quad -(3)$$

Apply (iii) in (3).

$$u(x,\infty) = B\sin\frac{n\pi x}{10}\left(ce^{\infty}+De^{-\infty}\right)$$

$$0 = B\sin\frac{n\pi x}{10}\,ce^{\infty}$$

$e^{\infty} \neq 0, \quad \sin\frac{n\pi x}{10} \neq 0, \quad B \neq 0 \Rightarrow \boxed{c=0}$

sub in (3).

$$\therefore u(x,y) = B\sin\frac{n\pi x}{10}\,De^{-\frac{n\pi y}{10}}$$

$BD = b_n$

∴ The most general solution is

$$u(x,y)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}} \quad — ④$$

Apply (iv) in ④.

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}\,(1) = \begin{cases} 20x, & 0\le x\le 5 \\ 20(10-x), & 5\le x\le 10 \end{cases}$$

Apply half range Sine Series,

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$=\frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx$$

$$=\frac{1}{5}\left[\int_0^5 20x\sin\frac{n\pi x}{10}\,dx+\int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$=\frac{20}{5}\left[\int_0^5 x\sin\frac{n\pi x}{10}\,dx+\int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$=4\left[\int_0^5 x\sin\frac{n\pi x}{10}\,dx+\int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$=4\left[x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)+\frac{\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5+\left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)-1\,\frac{\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}$$

$$=4\left[\frac{-50}{n\pi}\cancel{\cos\frac{n\pi}{2}}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]+\left[0-0+\frac{50}{n\pi}\cancel{\cos\frac{n\pi}{2}}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=4\,\frac{2(100)}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$B_n=\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore B_n=\begin{cases}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

sub in ④

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{800}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}}$$

[HW]

1. A rectangular plate with insulated surface is 8cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge $y=0$ is given by $u(x,0) = 100 \sin\frac{\pi x}{8}$ in $0 < x < 8$ while the two long edges $x=0$ and $x=8$ as well as the other short edges are kept at $0°c$, find the steady state temperature function $u(x,y)$.