Infinite plate:

1) An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x=0$ is kept at temperature given by,

$$u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10. \end{cases}$$

Find the steady state temperature distribution in the plate.

The Laplace eq is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

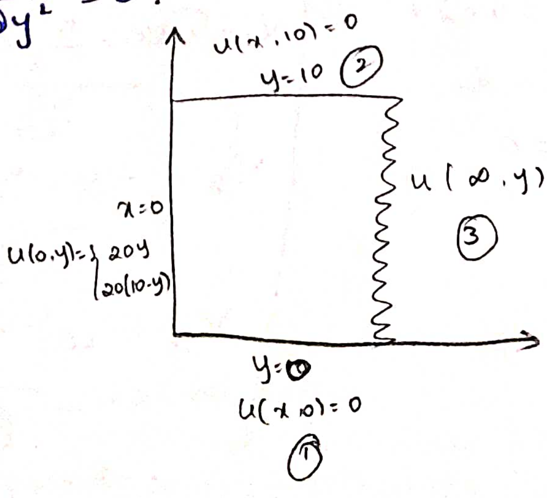
The boundary conditions are,

i) $u(x,0)=0$

ii) $u(x,10)=0$

iii) $u(\infty,y)=0$

iv) $u(0,y) = \begin{cases} 20y & ; 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$



~~The boundary conditions are,~~

The suitable solution is

$$u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y) \quad \text{— (1)}$$

Apply (i) in (1).

$$u(x,0) = (Ae^{\lambda x} + Be^{-\lambda x})(C) \Rightarrow 0 = C(Ae^{\lambda x} + Be^{-\lambda x})$$

$$\Rightarrow \boxed{C=0}, \quad Ae^{\lambda x} + Be^{-\lambda x} \neq 0$$

$$(1) \Rightarrow u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})\, D\sin\lambda y \quad \text{— (2)}$$

Apply (ii) in (2)

$$u(x,10) = (Ae^{\lambda x} + Be^{-\lambda x})\, D\sin 10\lambda.$$

$$0 = (Ae^{\lambda x} + Be^{-\lambda x})\, D\sin 10\lambda$$

$Ae^{\lambda x} + Be^{-\lambda x} \neq 0$, $D = 0$ [we get trivial soln].

$$\therefore \sin 10\lambda = 0 \Rightarrow 10\lambda = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{10}}$$

sub in (2).

$$u(x,y) = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right) D\sin\frac{n\pi y}{10} \quad \text{— (3)}$$

Apply (iii) in (3).

$$u(\infty,y) = (Ae^{\infty} + Be^{-\infty})\, D\sin\frac{n\pi y}{10}$$

$$0 = Ae^{\infty} D\sin\frac{n\pi y}{10}.$$

$e^{\infty} \neq 0$, $\sin\frac{n\pi y}{10} \neq 0$, $D \neq 0$ [$\because$ we get trivial soln]

$$\therefore \boxed{A=0}$$

sub in (3).

$$u(x,y) = Be^{-\frac{n\pi x}{10}} D\sin\frac{n\pi y}{10}.$$

$$= BDe^{-\frac{n\pi x}{10}}\sin\frac{n\pi y}{10} = B_n e^{-\frac{n\pi x}{10}}\sin\frac{n\pi y}{10} \quad [\because B_n = BD]$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi x}{10}}\sin\frac{n\pi y}{10} \quad \text{— (4)}$$

Apply (iv) in (4)

$$u(0,y) = \sum_{n=1}^{\infty} B_n(1)\sin\frac{n\pi y}{10} = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$$

Apply HRSS,

$$b_n = \frac{2}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy$$

$$= \frac{2}{10}\int_0^{10} f(y)\sin\frac{n\pi y}{l}\,dy = \frac{2}{10}\int_0^5 20y\sin\frac{n\pi y}{10}\,dy + \int_5^{10} 20(10-y)\sin\frac{n\pi y}{10}\,dy$$

$$= \frac{20}{5}\left[\int_0^5 y\sin\frac{n\pi y}{10}\,dy + \int_5^{10}(10-y)\sin\frac{n\pi y}{10}\,dy\right]$$

$$= 4\left[y\left(\frac{-\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right) + \frac{\sin\frac{n\pi y}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]_0^5 + \left[(10-y)\left(-\frac{\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right) - 1\cdot\frac{\sin\frac{n\pi y}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]_5^{10}$$

$$= 4\left[\cancel{\frac{-50}{n\pi}\cos\frac{n\pi}{2}} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \left[0 - 0 + \cancel{\frac{50}{n\pi}\cos\frac{n\pi}{2}} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= 4\,\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$B_n = \begin{cases} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

sub in (4) $\Rightarrow u(x,y) = \sum_{n=odd}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\, e^{\frac{-n\pi x}{10}}\sin\frac{n\pi y}{10}$