Difference Equations :

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Formation of difference Equations :

I. Form the difference eqn. from

i). $y_n = a + b\,3^n$

ii). $y_n = A2^n + Bn$

Soln. :

i). $y_n = a + b\,3^n \rightarrow (1)$

$y_{n+1} = a + b\,3^{n+1}$

$= a + b\,3^n . 3^1 \rightarrow (2)$

$y_{n+2} = a + b\,3^{n+2}$

$= a + b\,3^n\,9 \rightarrow (3)$

Eliminating a & b from (1), (2) and (3),

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n(9-3) - 1(9y_{n+1} - 3y_{n+2}) + 1(y_{n+1} - y_{n+2}) = 0$$

$$6y_n - 8y_{n+1} + 2y_{n+2} = 0 \quad (\div 2)$$

$$3y_n - 4y_{n+1} + y_{n+2} = 0$$

ii). $y_n = A2^n + Bn \longrightarrow (1)$

$y_{n+1} = A2^{n+1} + B(n+1)$

$y_{n+1} = A2^n \cdot 2 + B(n+1) \longrightarrow (2)$

$y_{n+2} = A2^{n+2} + B(n+2)$

$= 4A2^n + B(n+2) \longrightarrow (3)$

Eliminating 'A' & 'B' from (1), (2) and (3),

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & n+1 \\ y_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$y_n(2n+4-4n-4) - 1(ny_{n+1} + 2y_{n+1} - ny_{n+2} - y_{n+2})$

$+ n(4y_{n+1} - 2y_{n+2}) = 0$

$-2ny_n - ny_{n+1} - 2y_{n+1} + ny_{n+2} + y_{n+2} +$

$4ny_{n+1} - 2ny_{n+2} = 0$

$-2ny_n + 3ny_{n+1} - 2y_{n+1} - ny_{n+2} + y_{n+2} = 0$

$-2ny_n + (3n-2)y_{n+1} + (-n+1)y_{n+2} = 0$

HW J. Form the difference eqn. from

i). $y_n = (A+Bn)2^n$

ii). $y_n = A + B2^n$

Solving of Difference Equation :

Formulas :

$$Z[y_n] = F(z)$$

$$Z[y_{n+1}] = zF(z) - z\,y_0$$

$$Z[y_{n+2}] = z^2F(z) - z^2 y_0 - z y_1$$

$$Z[y_{n+3}] = z^3F(z) - z^3 y_0 - z^2 y_1 - z y_2$$

where $y_0 = y(0)$

$y_1 = y(1)$

$y_2 = y(2)$

I. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$ and $y_1 = 1$ using z-transform.

Soln. :

$$y_{n+2} + 4y_{n+1} + 3y_n = 2^n$$

Taking Z-transform on both sides,

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[2^n]$$

$$z^2F(z) - z^2 y_0 - z y_1 + 4[zF(z) - z y_0] + 3F(z) = \frac{z}{z-2}$$

$$z^2F(z) - 0 - z + 4zF(z) + 3F(z) = \frac{z}{z-2}$$

$$(z^2+4z+3)F(z) = \frac{z}{z-2} + z$$

$$(z^2+4z+3)F(z) = \frac{z + z^2 - 2z}{z-2}$$

$$\therefore F(z) = \frac{z^2 - z}{(z-2)(z^2+4z+3)}$$

$$F(z) = \frac{z^2 - z}{(z-2)(z+1)(z+3)}$$

i.e. $$F(z) = \frac{z(z-1)}{(z-2)(z+1)(z+3)}$$

$$\frac{F(z)}{z} = \frac{z-1}{(z-2)(z+1)(z+3)} \rightarrow (1)$$

$$\frac{z-1}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3}$$

$$= \frac{A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)}{(z-2)(z+1)(z+3)}$$

$$z-1 = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)$$

when $z=2$,

$1 = A(3)(5)$

$A = \frac{1}{15}$

when $z=-1$

$-2 = B(-3)(2)$

$B = 1/3$

when $z=-3$

$-4 = C(-5)(-2)$

$C = -2/5$

$$(1) \Rightarrow \frac{F(z)}{z} = \frac{1/15}{z-2} + \frac{1/3}{z+1} - \frac{2/5}{z+3}$$

$$F(z) = \frac{1}{15}\frac{z}{z-2} + \frac{1}{3}\frac{z}{z+1} - \frac{2}{5}\frac{z}{z+3}$$

Taking Z^{-1} on bothsides,

$$Z^{-1}[F(z)] = \frac{1}{15} Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{3} Z^{-1}\left[\frac{z}{z-(-1)}\right] - \frac{2}{5} Z^{-1}\left[\frac{z}{z-(-3)}\right]$$

$$y_n = \frac{1}{15}(2)^n + \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n$$

2]. Solve using Z transforms

$y_{n+2} - 3y_{n+1} - 10y_n = 0$ with $y_0 = 1,\ y_1 = 0$

Soln.:

$$y_{n+2} - 3y_{n+1} - 10y_n = 0$$

$$Z[y_{n+2}] - 3Z[y_{n+1}] - 10Z[y_n] = 0$$

$$z^2 F(z) - z^2 y_0 - z F(z) - 3[zF(z) - zy_0] - 10F(z) = 0$$

$$z^2 F(z) - z^2 - 0 - 3[zF(z) - z] - 10F(z) = 0$$

$$z^2 F(z) - z^2 - 3zF(z) + 3z - 10F(z) = 0$$

$$[z^2 - 3z - 10] F(z) = z^2 - 3z$$

$$F(z) = \frac{z(z-3)}{(z-5)(z+2)}$$

$$\therefore F(z) = \frac{z(z-3)}{(z-5)(z+2)}$$

$$\frac{F(z)}{z} = \frac{z-3}{(z-5)\ (z+2)} \rightarrow (1)$$

$$\frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2}$$

$$= \frac{A(z+2) + B(z-5)}{(z-5)(z+2)}$$

$$z-3 = A(z+2) + B(z-5)$$

when $z=-2$

$-5 = -7B$

$B = 5/7$

when $z=5$,

$2 = 7A$

$A = 2/7$

$$(1) \Rightarrow \quad \frac{F(z)}{z} = \frac{2/7}{z-5} + \frac{5/7}{z+2}$$

$$F(z) = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

Taking z^{-1} on bothsides,

$$z^{-1}[F(z)] = \frac{2}{7} z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7} z^{-1}\left[\frac{z}{z+2}\right]$$

$$y_n = \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n$$

3]. Solve the difference eqn. using Z-transform $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$

Soln. :

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$$

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$$z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$$

$$z^2 F(z) - z^2 y_0 - z y_1 + 6[zF(z) - z y_0] + 9F(z) = \frac{z}{z-2}$$

$$z^2 F(z) + 6z + 9F(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9) F(z) = \frac{z}{z-2}$$

$$F(z) = \frac{z}{(z-2)(z+3)^2}$$

$$F(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2} \longrightarrow (i)$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$= \frac{A(z+3)^2 + B(z-2)(z+3) + C(z-2)}{(z-2)(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

when $z = -3$	when $z = 2$	when $z = 0$
$1 = C(-5)$	$1 = A(5)^2$	$1 = 9A - 6B - 2C$
$C = -1/5$	$A = 1/25$	$+6B = 9A - 2C - 1$
		$6B = \frac{9}{25} + \frac{2}{5} - 1$

$$6B = \frac{9+10-25}{25}$$

$$B = \frac{-6}{25(6)}$$

$$B = \frac{-1}{25}$$

$$(iii) \Rightarrow \frac{F(z)}{z} = \frac{1/25}{z-2} + \frac{-1/25}{z+3} + \frac{-1/5}{(z+3)^2}$$

$$F(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$z^{-1}[F(z)] = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right]$$

$$+ \frac{1}{15} z^{-1}\left[\frac{z(-3)}{(z-(-3))^2}\right]$$

$$y_n = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15} n(-3)^n$$

4). Solve $y(n+2) - 4y(n+1) + 4y(n) = 0$ with $y(0) = 1,\ y(1) = 0$ using z-transform

Soln.

$$y(n+2) - 4y(n+1) + 4y(n) = 0$$

$$Z[y(n+2)] - 4Z[y(n+1)] + 4Z[y(n)] = 0$$

$$z^2F(z) - z^2y(0) - zy(1) - 4[zF(z) - zy(0)] + 4F(z) = 0$$

$$z^2F(z) - z^2 - 0 - 4[zF(z) - z] + 4F(z) = 0$$

$$z^2F(z) - 4zF(z) + 4F(z) - z^2 + 4z = 0$$

$$(z^2 - 4z + 4)F(z) = z^2 - 4z$$

$$F(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$F(z) = \frac{z(z-4)}{(z-2)^2}$$

$$\frac{F(z)}{z} = \frac{z-4}{(z-2)^2} \rightarrow (1)$$

$$\frac{z-4}{(z-2)^2} = \frac{A}{z-2} + \frac{B}{(z-2)^2}$$

$$\frac{z-4}{(z-2)^2} = \frac{A(z-2) + B}{(z-2)^2}$$

$$z - 4 = A(z-2) + B$$

When $z = 2$

$-2 = B$

$B = -2$

$z = 0$

$-4 = -2A + B$

$2A = B + 4$

$= -2 + 4 = 2$

$2A = 2$

$A = 1$

$$(1) \Rightarrow \frac{F(z)}{z} = \frac{1}{z-2} + \frac{-2}{(z-2)^2}$$

$$F(z) = \frac{z}{z-2} - 2\frac{z}{(z-2)^2}$$

$$z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-2}\right] - \cancel{2}\, z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$y_n = (2)^n - n(2)^n$$

$$z[na^n] = \frac{az}{(z-a)^2}$$

5]. Solve $y(n+3) - 3y(n+1) + 2y(n) = 0$ with $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$

Soln. :

$$y(n+3) - 3y(n+1) + 2y(n) = 0$$

$$z[y(n+3)] - 3z[y(n+1)] + 2z[y(n)] = 0$$

$$z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$$

$$-3[zF(z) - zy(0)] + 2F(z) = 0$$

$$z^3F(z) - 4z^3 - 0 - 8z - 3[zF(z) - 4z] + 2F(z) = 0$$

$$z^3F(z) - 4z^3 - 8z - 3zF(z) + 12z + 2F(z) = 0$$

$$z^3F(z) - 4z^3 + 4z = 0$$

$$(z^3 - 3z + 2)F(z) - 4z^3 + 4z = 0$$

$$(z^3 - 3z + 2)F(z) = 4z^3 - 4z$$

$$(z^3 - 3z + 2) \qquad F(z) = 4z(z^2-1)$$

1	1	0	−3	2
	0	1	1	−2
	1	1	−2	0

$$(z-1)(z^2+z-2) = 0$$

$$(z-1)(z-1)(z+2) = 0$$

$$\begin{array}{ccc} & -2 & \\ & \wedge & \\ 2 & & -1 = \end{array}$$

$$\therefore \; F(z) = \frac{4z(z^2-1)}{(z-1)^2(z+2)}$$

$$\frac{F(z)}{z} = \frac{4(z+1)(z-1)}{(z-1)^2(z+2)}$$

$$\frac{F(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)} \rightarrow (1)$$

Now $\frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$

$$= \frac{A(z+2) + B(z-1)}{(z-1)(z+2)}$$

$$4(z+1) = A(z+2) + B(z-1)$$

when $z=1$	$z=-2$
$4(2) = A(3)$	$4(-1) = B(-3)$
$A = 8/3$	$B = 4/3$

$$(1) \Rightarrow \frac{F(z)}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$F(z) = \frac{8}{3}\,\frac{z}{z-1} + \frac{4}{3}\,\frac{z}{z+2}$$

$$Z^{-1}[F(z)] = \frac{8}{3} Z^{-1}\left[\frac{z}{z-1}\right] + \frac{4}{3} Z^{-1}\left[\frac{z}{z-(-2)}\right]$$

$$y(n) = \frac{8}{3}(1) + \frac{4}{3}(-2)^n$$

$$y(n) = \frac{8}{3} + \frac{4}{3}(-2)^n$$