Inverse Z-transform:

If $z[f(n)] = F(z)$, then $f(n)$ is called the inverse Z-transform of $F(z)$ and is denoted by

$$f(n) = Z^{-1}\{F(z)\}$$

Convolution Theorem:

If $F(z)$ and $G(z)$ are the Z-transforms of $f(n)$ and $g(n)$ respectively then

$Z\{f(n) * g(n)\} = F(z)\,G(z)$ where $f(n) * g(n)$ is defined as the convolution of $f(n)$ and $g(n)$ given by $f(n) * g(n) = \sum_{k=0}^{n} f(k)\,g(n-k)$

Proof:-

We have $F(z)G(z) = \left[\sum_{n=0}^{\infty} f(n)\,z^{-n}\right]\left[\sum_{n=0}^{\infty} g(n)\,z^{-n}\right]$

$$F(z)G(z) = \left[f(0) + f(1)z^{-1} + \dots + f(n)z^{-n} + \dots\right]$$
$$\left[g(0) + g(1)z^{-1} + \dots + g(n)z^{-n}\right]$$

$$= \sum_{n=0}^{\infty}\left[f(0)g(n) + f(1)g(n-1) + \dots + f(n)g(0)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} A_n z^{-n}$$

where $A_n = f(0)g(n) + \dots + f(n)g(0)$

$$= \sum_{k=0}^{n} f(k)\,g(n-k) = f(n) * g(n)$$

$$F(z)\,G(z) = \sum_{n=0}^{\infty} A_n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(f(n) * g(n)\right) z^{-n}$$

$$F(z)\,G(z) = Z\{f(n) * g(n)\}$$

Inverse z-transform using convolution theorem.

1. Using convolution theorem, find $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$

WKT $z[a^n] = \frac{z}{z-a} \Rightarrow z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Let $F(z) = \frac{z}{z-a}$ and $G(z) = \frac{z}{z-b}$

$$\therefore\ z^{-1}[F(z)\,G(z)] = z^{-1}[F(z)] * z^{-1}[G(z)]$$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = a^n * b^n$$

$$= \sum_{k=0}^{n} a^k\, b^{n-k}$$

$$= b^n \sum_{k=0}^{n} a^k\, b^{-k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{-1 + \left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right) - 1}\right] = b^n\left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

(1) Find the inverse z-transform of $\frac{z}{(z+1)^2}$ by power series method.

Let $F(z) = \frac{z}{(z+1)^2} = \frac{z}{z^2\left(1+\frac{1}{z}\right)^2} = \frac{1}{z}\left[1+\frac{1}{z}\right]^{-2}$

$$F(z) = \frac{1}{z}\left[1 - \frac{2}{z} + \frac{3}{z^2} - \frac{4}{z^3} + \dots\right]$$

$$= \frac{1}{z} - \frac{2}{z^2} + \frac{3}{z^3} - \frac{4}{z^4} + \dots$$

$$= z^{-1} - 2z^{-2} + 3z^{-3} - 4z^{-4} + \dots$$

$$Z\{f(n)\} = \sum_{n=0}^{\infty} n(-1)^{n+1} z^{-n}.$$

$$Z\{f(n)\} = Z\left[n(-1)^{n+1}\right]$$

$$f(n) = n(-1)^{n+1}$$