Properties :-

1. Linear Property:

$$z\left[af(n)+bg(n)\right] = aF(z)+bG(z) \quad , \quad z\left[f(n)\right]=F(z)$$
$$z\left[g(n)\right]=G(z)$$

'a' and 'b' are constants

Proof :-

$$z\left[af(n)+bg(n)\right] = \sum_{n=0}^{\infty}\left[af(n)+bg(n)\right]z^{-n}$$
$$= \sum_{n=0}^{\infty} af(n)z^{-n} + \sum_{n=0}^{\infty} b\,g(n)z^{-n}$$
$$= a\,z\left[f(n)\right] + b\,z\left[g(n)\right]$$
$$= aF(z)+bG(z)$$

2. First Shifting Property :-

If $z\left[f(t)\right]=F(z)$, then $z\left[e^{-at}f(t)\right]=F\left[ze^{aT}\right]$

(or)

$$z\left[e^{-at}f(t)\right] = \left\{F(z)\right\}_{z\to ze^{aT}} \quad , \quad z\left\{e^{at}f(t)\right\} = \left\{F(z)\right\}_{z\to ze^{-aT}}$$

Proof :-

$$z\left[f(t)\right] = \sum_{n=0}^{\infty} f(nT)z^{-n}$$
$$z\left[e^{-at}f(t)\right] = \sum_{n=0}^{\infty} e^{-anT}f(nT)z^{-n}$$
$$= \sum_{n=0}^{\infty} f(nT)\left(ze^{aT}\right)^{-n}$$
$$= F\left[ze^{aT}\right]$$

3. Change of Scale:

If $z[f(n)] = F(z)$ then $z[a^n f(n)] = F\left(\frac{z}{a}\right)$

Proof:- $z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$\Rightarrow z[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n) z^{-n} = \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n}$

$= F\left(\frac{z}{a}\right) = \{F(z)\}_{z \to \frac{z}{a}}.$

4. Second Shifting Property:

If $z[f(n)] = F(z)$ then $z[f(n+1)] = zF(z) - zf(0)$

5). Differentiation in z-domain:

i) $z[nf(n)] = -z \frac{d}{dz}\{F(z)\}$, where $F(z) = z\{f(n)\}$

ii) $z[nf(t)] = -z \frac{d}{dz}\{f(z)\}$, where $F(z) = z[f(t)]$

Proof:- $F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$

$\frac{d}{dz} F(z) = \frac{d}{dz} \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} f(n)(-n) z^{-n-1}$

$= \frac{-1}{z} \sum_{n=0}^{\infty} n f(n) z^{-n} = \frac{-1}{z} z[nf(n)]$

$z[nf(n)] = -z \frac{d}{dz} F(z)$

1. Find $z[e^{-iat}]$

$$z[e^{-iat}] = z[e^{-iat}(1)] = [z(1)]_{z \to ze^{iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{iaT}}$$

$$= \frac{ze^{iaT}}{ze^{iaT}-1}$$

2. Find $z[\cos at]$ and $z[\sin at]$

$$z[e^{iaT}] = z[e^{iat}(1)] = [z(1)]_{z \to ze^{-iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{-iaT}}$$

$$= \frac{ze^{-iaT}}{ze^{-iaT}-1}$$

$\div$ by e^{-iaT}

$$= \frac{ze^{-iaT}/e^{-iaT}}{\frac{ze^{-iaT}-1}{e^{-iaT}}} = \frac{z}{z-e^{iaT}}$$

$$z[\cos at + i\sin at] = \frac{z}{z-(\cos aT + i\sin aT)}$$

$$= \frac{z}{(z-\cos aT)-i\sin aT} \times \frac{(z-\cos aT)+i\sin aT}{(z-\cos aT)+i\sin aT}$$

$$= \frac{z(z-\cos aT)+iz\sin aT}{(z-\cos aT)^2+(\sin aT)^2}$$

$$z\cos aT + iz\sin aT = \frac{z(z-\cos aT) + iz\sin aT}{z^2 - 2z\cos aT + 1}$$

Equating real & imaginary parts we get,

$$Z[\cos aT] = \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1},$$

$$Z[\sin aT] = \frac{z\sin at}{z^2 - 2z\cos aT + 1}$$