# One dimensional Heat Equation [parabolic]

## One DHE [Rod]

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where $a^2 = \dfrac{\text{Thermal conductivity}}{\text{Density} \times \text{Specific Heat}}$

ie) $a^2 = \dfrac{K}{\rho C}$ which is called diffusivity

Possible Solutions of ODHE

1) $u(x,t) = (A_1 e^{px} + A_2 e^{-px}) A_3 e^{a^2 p^2 t}$

2) $u(x,t) = (A_4 \cos px + A_5 \sin px) A_6 e^{-a^2 p^2 t}$

3) $u(x,t) = (A_7 x + A_8) A_9$

Suitable solution:

$$u(x,t) = (A \cos px + B \sin px)\, e^{-a^2 p^2 t}$$

Assumptions of deriving the ODHE!

1. Heat flows from higher to lower temperature
2. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body, and to the temperature change.
3. The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the curve.

This constant of proportionality is known as the thermal conductivity (K) of the material. It is known as fourier's law of heat conduction.

Steady state condition:-

The state in which the temperature depends only on the distance but not on time t, is called steady state.

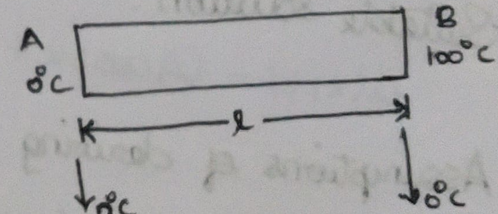
Therefore $u(x,t)$ becomes $u(x)$ under the steady state.

Note: $u(x) = \left(\frac{b-a}{l}\right)x + a.$

Type 1: [Steady state conditions bothends at zero temperature]

1. A rod of length $l$ has its ends A and B kept at $0°c$ and $100°c$ until steady state condition prevail. If the temperature at B is reduced suddenly to $0°c$ and kept so, while that of A is maintained, find the temperature $u(x,t)$ at a distance $x$ from A and at time t.

Solution :-




The ODHE is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

i) $u(0,t) = 0$

ii) $u(l,t) = 0$

iii) $u(x,0) = \frac{100x}{l}$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{100-0}{l}\right)x + 0$$

$$= \frac{100x}{l}$$

The suitable solution is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \rightarrow ①$$

Applying i) in ① we get $u(0,t) = 0$

$$(A + 0)\, e^{-a^2p^2t} = 0.$$

$$A\, e^{-a^2p^2t} = 0$$

Here $e^{-a^2p^2t} \neq 0$ [It is a function of time]

$$\Rightarrow A = 0$$

Sub $A = 0$ in ① we get

$$u(x,t) = B \sin px\, e^{-a^2p^2t} \rightarrow ②$$

Applying ii) in ② we get

$$u(l,t) = 0$$

$$B \sin pl\, e^{-a^2p^2t} = 0$$

Here $e^{-a^2p^2t} \neq 0$ [It is a function of 't']

$B \neq 0$ [Suppose $B = 0$, we get a trivial soln]

$$\Rightarrow \sin pl = 0$$

$$pl = \sin^{-1} 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Sub $p = \frac{n\pi}{l}$ in ②

$$u(x,t) = B \sin \frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2}{l^2}t} \rightarrow ③$$

The most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2}{l^2}t} \rightarrow ④$$

Applying iii) in ④

$$u(x,0) = \frac{100x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{100x}{l}$$

Half Range sine series

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{100x}{l} \qquad \therefore B_n = b_n$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\left[x\frac{-\cos\frac{n\pi x}{l}}{n\pi/l} - 1\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{-lx}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n\right]$$

$$= \frac{-200(-1)^n}{n\pi}$$

$$B_n = \frac{200(-1)^{n+1}}{n\pi}$$

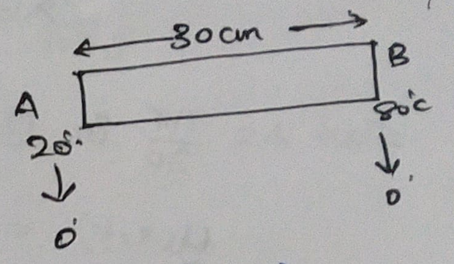
Sub $B_n$ value in (4), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2t}{l^2}}$$

$$= \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{a^2n^2\pi^2t}{l^2}}$$

2. A rod 30 cm long has its ends A and B kept at 20° and 80° respectively. Until steady state conditions prevails. The temperature at each end is then suddenly reduced to 0°c and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.

Solution:




The ODHE is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{80-20}{30}\right)x + 20$$

$$= \left(\frac{60}{30}\right)x + 20$$

$$= 2x + 20$$

The boundary conditions are

i) $u(0,t) = 0, \; t > 0$

(ii) $u(30,t) = 0, \; t > 0$

(iii) $u(x,0) = 2x + 20$

The suitable soln is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-a^2\lambda^2 t} \rightarrow ①.$$

Applying (i) in ①

$u(0,t) = 0.$

$$u(0,t) \Rightarrow (A(1) + B(0))\, e^{-a^2\lambda^2 t} = 0.$$

$$A\, e^{-a^2\lambda^2 t} = 0.$$

$e^{-a^2\lambda^2 t} \neq 0$ [since it is a fn of time]

$\boxed{A = 0}$

Sub $A=0$ in ①

$$u(x,t) = B\sin\lambda x\, e^{-a^2\lambda^2 t} \rightarrow ②.$$

Apply (ii) in ②

$u(30,t) = 0.$

$$u(30,t) = B\sin\lambda 30\, e^{-a^2\lambda^2 t} = 0$$

Here $e^{-a^2\lambda^2 t} \neq 0$ [since it is a fn of time]

and $B \neq 0$. [Suppose $B = 0$, we get trivial solution]

$$\sin \lambda 30 = 0$$
$$\lambda(30) = \sin^{-1} 0$$
$$\lambda(30) = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{30}}$$

Sub $\lambda = \frac{n\pi}{30}$ in ②

$$u(x,t) = B \sin \frac{n\pi x}{30} \, e^{-a^2 \frac{n^2\pi^2}{900} t}$$
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} \, e^{-a^2 \frac{n^2\pi^2}{900} t}$$

Apply (iii) in ③

$u(x,0) = f(x)$ in ③

$$= 2x + 20$$

$$u(x,0) \Rightarrow \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} \, e^{0}$$
$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}$$

$$\therefore \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} = 2x + 20$$

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30} = 2x + 20 \qquad B_n = b_n$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

$$\therefore b_n = \frac{2}{30} \int_0^{30} (2x + 20) \sin \frac{n\pi x}{30} \, dx$$

$$= \frac{1}{15} \left[ (2x+20) \left( \frac{-\cos \frac{n\pi x}{30}}{n\pi/30} \right) - (2) \left( \frac{-\sin \frac{n\pi x}{30}}{(n\pi/30)^2} \right) \right]_0^{30}$$

$$= \frac{1}{15}\left\{\left[2(30)+20\right]\left[\frac{-\cos\frac{n\pi 30}{30}}{\frac{n\pi}{30}}\right] + 2\left[\frac{\sin\frac{n\pi 30}{30}}{n^2\pi^2/900}\right]\right.$$

$$\left. - \left\{30\left(\frac{-\cos 0}{n\pi/30}\right) + 2\left(\frac{\sin 0}{n^2\pi^2/900}\right)\right\}\right]$$

$$= \frac{1}{15}\left[-80(-1)^n\left(\frac{30}{n\pi}\right) + 20(1)\frac{30}{n\pi}\right]$$

$$= \frac{1}{15}\left[-\frac{2400}{n\pi}(-1)^n + \frac{600}{n\pi}\right]$$

$$= \frac{1}{15}\left[\frac{600}{n\pi}\right]\left[1-4(-1)^n\right]$$

$$= \frac{40}{n\pi}\left[1-4(-1)^n\right] \rightarrow 4$$

$$4 \Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[1-4(-1)^n\right] \sin\frac{n\pi x}{30}\, e^{-a^2\frac{n^2\pi^2}{900}t}$$