Problems on convolution

3. find the z-transform of $f(n) * g(n)$ where $f(n) = \left(\frac{1}{2}\right)^n$ and $g(n) = \cos n\pi$ using convolution theorem.

By convolution theorem,

$$Z[f(n) * g(n)] = Z[f(n)] \cdot Z[g(n)] \rightarrow ①$$

$$Z[f(n)] = Z\left[\left(\frac{1}{2}\right)^n\right] = \frac{z}{z - \frac{1}{2}} = \frac{2z}{2z-1}$$

and $Z[g(n)] = Z[\cos n\pi]$

$$= Z[(-1)^n] = \frac{z}{z-(-1)} = \frac{z}{z+1}$$

$$① \Rightarrow \quad Z[f(n) * g(n)] = \frac{2z}{2z-1}\left(\frac{z}{z+1}\right)$$

$$= \frac{2z^2}{(2z-1)(z+1)}$$

4. Using Convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left(\frac{z}{(z-1)} \cdot \frac{z}{z-3}\right)$$

$$= Z^{-1}\left(\frac{z}{z-1}\right) * Z^{-1}\left(\frac{z}{z-3}\right)$$

$$= (1)^n * 3^n$$

$$= \sum_{k=0}^{n} (1)^k \cdot 3^{n-k}$$

$$= \sum_{k=0}^{n} 3^{n-k}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + 3^{n-3} + \dots + 3^0$$

$= 1 + 3 + \dots + 3^{n-2} + 3^{n-1} + 3^{n}$

$= \dfrac{3^{n+1} - 1}{3 - 1}$

$= \dfrac{3^{n+1} - 1}{2}.$

5. Using convolution theorem, find $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{8z^2}{2(z-1/2).4(z+1/4)}\right]$$

$$= Z^{-1}\left[\frac{z^2}{(z-1/2)(z+1/4)}\right]$$

$$= Z^{-1}\left[\frac{z}{(z-\frac{1}{2})}\cdot\frac{z}{(z+\frac{1}{4})}\right]$$

$$= Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) * Z^{-1}\left(\frac{z}{z+\frac{1}{4}}\right)$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{-1}{4}\right)^n$$

$$= \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K \left(\frac{-1}{4}\right)^{n-K}$$

$$= \left(\frac{-1}{4}\right)^n \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K (-4)^K$$

$$= \left(\frac{-1}{4}\right)^n \sum_{K=0}^{n} \left[(-4)\left(\frac{+1}{2}\right)\right]^K = \left(\frac{-1}{4}\right)^n \sum_{K=0}^{n} (-2)^K$$

$$= \left(\frac{-1}{4}\right)^n \left[1 + (-2) + (-2)^2 + \dots + (-2)^n\right]$$

$$= \left(\frac{-1}{4}\right)^n \left[\frac{1-(-2)^{n+1}}{1-(-2)}\right] = \left(\frac{-1}{4}\right)^n \left[\frac{1+2(-2)^n}{3}\right]$$

6. Using Convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-4}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 4^n * 3^n$$

$$= \sum_{K=0}^{n} 4^K \cdot 3^{n-K}$$

$$= 3^n \sum_{K=0}^{n} 4^K \cdot 3^{-K} = 3^n \sum_{K=0}^{n} \left(\frac{4}{3}\right)^K$$

$$= 3^n\left[\left(\frac{4}{3}\right)^0 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^n\right]$$

$$= 3^n\left[1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^n\right]$$

$$= 3^n\left[\frac{(4/3)^{n+1} - 1}{4/3 - 1}\right]$$

7) Use Convolution, find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right] = Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{K=0}^{n} a^K \cdot a^{n-K}$$

$$= \sum_{K=0}^{n} a^n$$

$$= (n+1)a^n.$$