Inverse z-transform by using Partial fraction Method.

1. Find the inverse z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$

Let $F(z) = \frac{z^2+z}{(z-1)(z^2+1)} = \frac{Az}{z-1} + \frac{Bz^2+Cz}{z^2+1}$

$$z^2+z = Az(z^2+1) + (Bz^2+Cz)(z-1)$$

Put $z=1$

$2 = 2A$

$\boxed{A=1}$

Put coeff of z^3

$A+B=0$

$1+B=0$

$\boxed{B=-1}$

Coeff of z

$1 = A - C$

$1 = 1 - C$

$\boxed{C=0}$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

$$Z^{-1}[F(z)] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$= 1^n - \cos\frac{n\pi}{2}$$

$$= 1 - \cos\frac{n\pi}{2}.$$

2. Find the inverse Z-transform of $\frac{z^3}{(z-1)^2(z-2)}$

Let $F(z) = \frac{z^3}{(z-1)^2(z-2)}$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2} \rightarrow (1)$$

$$\frac{z^2}{(z-1)^2(z-2)} = \frac{A(z-1)(z-2)+B(z-2)+C(z-1)^2}{(z-1)^2(z-2)}$$

$$z^2 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

When $z=1$, $\Rightarrow$ $1=-B$ $\Rightarrow \boxed{B=-1}$

$z=2 \Rightarrow$ $4=C$ $\Rightarrow \boxed{C=4}$

$z=0 \Rightarrow$ $2A+(-2B)+C=0$

$2A+2+4=0$ $\Rightarrow 2A=-6$

$\boxed{A=-3}$

$$(1) \Rightarrow \frac{F(z)}{z} = \frac{-3}{z-1} + \frac{-1}{(z-1)^2} + \frac{4}{z-2}$$

$$\frac{z^3}{(z-1)^2(z-2)} = -3\frac{z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}$$

$$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = -3\,Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4\left[\frac{z}{z-2}\right]$$

$$= -3(1) - n + 4(2)^n$$

$$= 4\cdot 2^n - 3 - n$$

$$\frac{z}{z-a} = Z(a^n)$$

$$Z(1) = \frac{z}{z-1}$$

$$\frac{z}{(z-1)^2} = Z(n)$$

8. Find $Z^{-1}\left[\frac{10z}{z^2-3z+2}\right]$

Let $F(z) = \frac{10z}{z^2-3z+2}$

$$\frac{F(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\frac{10}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$10 = A(z-2) + B(z-1)$$

When $z=1$, $\boxed{A=-10}$

$z=2$, $\boxed{B=10}$

$$\therefore \frac{F(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$F(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$Z^{-1}[F(z)] = -10\, Z^{-1}\left[\frac{z}{z-1}\right] + 10\, Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -10(1) + 10(2)^n$$

2. Find $Z^{-1}\left[\frac{z^2-3z}{(z-5)(z+2)}\right]$

Let $\frac{F(z)}{z} = \frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2} \rightarrow (1)$

$$\frac{z-3}{(z-5)(z+2)} = \frac{A(z+2) + B(z-5)}{(z-5)(z+2)}$$

When $z = -2 \Rightarrow -2-3 = B(-2-5)$

$$-5 = -7B$$

$$\boxed{B = 5/7}$$

$z = 5 \Rightarrow 5-3 = A(5+2)$

$$2 = 7A$$

$$\boxed{A = 2/7}$$

$① \Rightarrow \quad \frac{F(z)}{z} = \frac{2/7}{z-5} + \frac{5/7}{z+2}$

$$F(z) = \frac{2}{7} \cdot \frac{z}{z-5} + \frac{5}{7} \cdot \frac{z}{z+2}$$

$$z^{-1}[F(z)] = \frac{2}{7} z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7} z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n.$$

Method of residue.

$f(n) =$ Sum of the residues of $z^{n-1}f(z)$ at its poles.

Note:

1. The pole of order 1 at $z=a$ is

$$\{\text{Res } z^{n-1}F(z)\}_{z=a} = \lim_{z\to a} (z-a)\left[z^{n-1}F(z)\right]$$

2. The pole of order m at $z=a$ is,

$$\{\text{Res } z^{n-1}F(z)\}_{z=a} = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} (z-a)^m z^{n-1}F(z)$$

Problems on Cauchy's Residue method

1. Find $z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

Let $F(z) = \frac{z}{(z-1)(z-2)}$

$$z^{n-1}F(z) = z^{n-1}\cdot\frac{z}{(z-1)(z-2)} = \frac{z^n}{(z-1)(z-2)}$$

Here $z=1$ is a pole of order 1

and $z=2$ is a pole of order 1

$$\text{Res}\{z^{n-1}F(z)\}_{z=1} = \lim_{z\to 1} (z-1)\frac{z^n}{(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{z^n}{z-2}$$

$$= \frac{(1)^n}{1-2} = -(1)^n = -1$$

$$\text{Res}\{z^{n-1}F(z)\}_{z=2} = \lim_{z\to 2} (z-2)\frac{z^n}{(z-1)(z-2)} = \lim_{z\to 2}\frac{z^n}{z-1}$$

$$= 2^n$$

$\therefore f(n) =$ Sum of the residues of $z^{n-1} F(z)$ at its pole.

$= -1 + 2^n$

d) Find $Z^{-1}\left[\dfrac{z(z+1)}{(z-1)^2}\right]$

Let $F(z) = \dfrac{z(z+1)}{(z-1)^2}$

$$z^{n-1} F(z) = z^{n-1} \frac{z(z+1)}{(z-1)^2}$$

$$= \frac{z^n (z+1)}{(z-1)^2}$$

Here $z=1$ is a pole of order 2.

$$\underset{z=1}{\text{Res}} \{z^{n-1} F(z)\} = \lim_{z \to 1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[(z-1)^2 \frac{z^n (z+1)}{(z-1)^2}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz} [z^n . (z+1)]$$

$$= \lim_{z \to 1} [z^n (1) + (z+1) n z^{n-1}]$$

$$= 1^n + (1+1)(n)(1)^{n-1}$$

$$= 1^n + 2n (1)^{n-1}$$

$$= 1 + 2n$$

$\therefore f(n) =$ Sum of the residues

$= 1 + 2n.$

8) Find $Z^{-1}\left[\frac{z^2}{z^2+4}\right]$ using Cauchy's Residue method

HW $Z^{-1}\left[\frac{z}{(z-1)(z^2+1)}\right]$

Let $F(z) = \frac{z^2}{z^2+4}$

$$z^{n-1}F(z) = z^{n-1}\frac{z^2}{z^2+4}$$

$$= \frac{z^{n+1}}{z^2+4}$$

$z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\pm 2i$

$z=2i$ is a pole of order 1

$z=-2i$ is a pole of order 1

$$\text{Res}\left[z^{n-1}F(z)\right]_{z=2i} = \lim_{z\to 2i}(z-2i)\frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i}\frac{z^{n+1}}{(z+2i)} = \frac{(2i)^{n+1}}{4i}$$

$$= \frac{(2i)^n(2i)}{4i} = \frac{(2i)^n}{2} = 2^{n-1}i^n.$$

$$\text{Res}\left[z^{n-1}F(z)\right]_{z=-2i} = \lim_{z\to -2i}(z-(-2i))\frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i}\frac{z^{n+1}}{(z-2i)}$$

$$= \frac{(-2i)^{n+1}}{-4i} = \frac{(-2i)^n(-2i)}{-4i}$$

$$= \frac{(-2i)^n}{2} = 2^{n-1}(-i)^n.$$

$\therefore f(n) =$ Sum of the Residues.

$$= 2^{n-1}(i)^n + 2^{n-1}(-i)^n = 2^{n-1}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

$$\Downarrow$$

$$2^{n-1}\left[2\cos\frac{n\pi}{2}\right]$$

$$= 2^n\cos\frac{n\pi}{2}.$$