1. A string is stretched & fastened to two points $x=0$ & $x=l$ apart motion is started by displacing the string into the form $y = K(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at time t.

One dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{where } a^2 = \frac{T}{M}$$

∴ The suitable solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \rightarrow ①$$

Boundary condition:

i) $y(0,t) = 0, \; t > 0$

ii) $y(l,t) = 0, \; t > 0$

Initial condition:-

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \quad 0 < x < l$

iv) $y(x,0) = f(x), \; 0 < x < l$

$= K(lx - x^2), \; 0 < x < l.$

Applying condition i) in eqn ①

$y(0,t) = (A\cos\lambda(0) + B\sin\lambda(0))(C\cos\lambda at + D\sin\lambda at)$

$0 = A(C\cos\lambda at + D\sin\lambda at)$

$A = 0$ [since Boundary condition]

$C\cos\lambda at + D\sin\lambda at \neq 0$

Applying $A=0$ in eqn ①

$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \rightarrow ②$

Applying condition ii) in eqn ②

$y(l,t) = B\sin\lambda l\,(c\cos\lambda at + D\sin\lambda at)$

$0 = B\sin\lambda l\,(c\cos\lambda at + D\sin\lambda at)$

$B \neq 0,\quad \sin\lambda l = 0,\quad c\cos\lambda at + D\sin\lambda at \neq 0.$

↳ only one constant should be zero(0)

$\sin\lambda l = 0$

$\lambda l = \sin^{-1}(0)$

$\lambda l = n\pi$ $\quad \boxed{\lambda = \frac{n\pi}{l}}$

Applying λ value in eqn ②

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(c\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow ③$$

Diff pwrt 't',

$$\frac{\partial y}{\partial t}(x,t) = B\sin\frac{n\pi x}{l}\left(-c\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l} + D\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)$$

when $t=0$,

$$\frac{\partial y}{\partial t}(x,0) = B\sin\frac{n\pi x}{l}\left(-c\sin\frac{n\pi a(0)}{l}\cdot\frac{n\pi a}{l} + D\cos\frac{n\pi a(0)}{l}\cdot\frac{n\pi a}{l}\right)$$

$$0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$B\sin\frac{n\pi x}{l} \neq 0,\quad D = 0,\quad \frac{n\pi a}{l} \neq 0.$

$$y(x,t) = B\sin\frac{n\pi x}{l}\cdot c\cos\frac{n\pi at}{l}$$

$$= Bc\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l}$$

$\boxed{Bc = b_n}$

$$y(x,t) = b_n' \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \rightarrow (4)$$

Applying condition (iv) in equation (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a(0)}{l}$$

$$K(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Expand b_n in Half range series,

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l}\int_0^l K(lx - x^2) \sin\frac{n\pi x}{l} \, dx$$

$$= \frac{2K}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l} \, dx$$

By applying Bernoulli's formula,

$u = lx - x^2$ $\qquad v = \sin\frac{n\pi x}{l}$

$u' = l - 2x$ $\qquad v_1 = \dfrac{-\cos\frac{n\pi x}{l}}{n\pi/l}$

$u'' = -2$

$v_2 = \dfrac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$, $v_3 = \dfrac{\cos\frac{n\pi x}{l}}{(n\pi/l)^3}$

$$b_n = \frac{2K}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^3}\right)\right]_0^l.$$

$$= \frac{2K}{l}\left[\left(0+0-2\frac{\cos n\pi}{(n\pi/l)^3}\right)-\left(0+0-\frac{2\cos 0}{(n\pi/l)^3}\right)\right]$$

$$= \frac{2K}{l}\left[\frac{-2(-1)^n}{(n\pi/l)^3}+\frac{2}{(n\pi/l)^3}\right] = \frac{4K}{l}\left(\frac{l^3}{n^3\pi^3}\right)\left[-(-1)^n+1\right]$$

$$= \frac{4K}{l} \times \frac{l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4Kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} \dfrac{8Kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

applying b_n in equation (4)

$$y(x,t) = \sum_{n=\text{odd}} \frac{8Kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} \quad \to (5).$$

2. A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3\frac{\pi x}{l}$ If it is released from rest from this position, then find the displacement.

The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

i) $y(0,t)=0, \ \forall t$ ii) $y(l,t)=0, \forall t$

iii) $\frac{\partial}{\partial t}y(x,0) = 0, \forall x$

iv) $y(x,0) = y_0 \sin^3\frac{\pi x}{l}.$

The suitable solution is

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \to ①$

Applying (i) in ① we get

$y(0,t) = 0$

$(A(1) + B(0))(C\cos pat + D\sin pat) = 0$

$A(C\cos pat + D\sin pat) = 0$

$C\cos pat + D\sin pat \neq 0$ (It is a function of time)

$\Rightarrow A = 0.$

Sub $A = 0$ in ①

$y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \to ②$

Applying (ii) in ② $y(l,t) = 0$

$B\sin pl\,(C\cos pat + D\sin pat) = 0$

$B \neq 0$ (If $B = 0$, we get a trivial solution)

$C\cos pat + D\sin pat \neq 0$ ($\because$ It is a function of 't')

$\Rightarrow \sin pl = 0.$

$pl = \sin^{-1}(0) \quad \Rightarrow pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

Sub $p = \frac{n\pi}{l}$ in ②

$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right] \to ③$

Before applying (iii) diff ③ p wrt 't'

$\frac{\partial}{\partial t} y(x,t) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$

Applying (iii) we get,

$$\frac{\partial}{\partial t} y(x,0) = 0$$

$$B \sin \frac{n\pi x}{l} \left[0 + D \frac{n\pi a}{l} \right] = 0.$$

$$BD \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 0.$$

Here $B \neq 0$ [If $B = 0$, we get a trivial soln]

$\sin \frac{n\pi x}{l} \neq 0$ [$\because$ It is a function of x]

$$\Rightarrow D = 0$$

Sub $D=0$ in (3)

$$y(x,t) = B \sin \frac{n\pi x}{l} \; C \cos \frac{n\pi at}{l}$$

$$= BC \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \rightarrow (4)$$

Applying condition (iv) in (4) we get

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \frac{y_0}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right]$$

$$\Rightarrow B_1 \sin \frac{\pi x}{l} + B_2 \sin \frac{2\pi x}{l} + B_3 \sin \frac{3\pi x}{l} + \ldots$$

$$= \frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l}$$

Equating the like coeff we get,

$$B_1 = \frac{3y_0}{4} \; ; \; B_2 = 0 \; ; \; B_3 = \frac{-y_0}{4} \; ; \; B_4 = B_5 = \ldots = 0.$$

Sub the above values in (4)

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}$$