Z-Transforms

Definition: [2 sided or bilateral]

Let $\{f(n)\}$ be a sequence defined $\forall$ integers then its z-transform is defined to be

$$F(z) = Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$, z is an arbitrary complex no.

Definition: [1-sided or unilateral]

Let $\{f(n)\}$ be a sequence defined $\forall$ positive integers then its z-transform is defined to be

$$F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Definition: z-transform for discrete values of 't':

If $f(t)$ is a function defined for discrete values of 't' where, $t = nT$, $n = 0, 1, 2, \ldots$, T being the sampling period, then z-transform of $f(t)$ is defined as

$$F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

Inverse z-transform:

If $Z[f(n)] = F(z)$, then $Z^{-1}[F(z)] = f(n)$ is called the inverse z-transform of $F(z)$.

Formulae:

1) $Z[1] = \dfrac{z}{z-1}$

2) $Z[a^n] = \dfrac{z}{z-a}$

3) $Z[(-a)^n] = \dfrac{z}{z+a}$

4) $Z[n] = \dfrac{z}{(z-1)^2}$

5) $Z[na^n] = \dfrac{az}{(z-a)^2}$

6) $Z[n^2] = \dfrac{z(z+1)}{(z-1)^3}$

7) $z\left[n a^{n-1}\right] = \frac{z}{(z-a)^2}$

8) $z\left[a^{n-1}\right] = \frac{1}{z-a}$

9) $z\left[(-a)^{n-1}\right] = \frac{1}{z+a}$

10) $Z\left[n(-a)^{n-1}\right] = \frac{Z}{(Z+a)^2}$

11) $z\left[(n-1)\, a^{n-2}\right] = \frac{1}{(z-a)^2}$

12) $z\left[(n-1)\,(-a)^{n-2}\right] = \frac{1}{(z+a)^2}$

13) $z\left[\cos \frac{n\pi}{2}\right] = \frac{z^2}{z^2+1}$

14) $z\left[\sin \frac{n\pi}{2}\right] = \frac{z}{z^2+1}$

15) $z\left[\frac{1}{n!}\right] = e^{1/z}$

1) Find the z-transform of $f(n)=1$ (or) $z[1]$.

(or)

PT $z[1] = \frac{z}{z-1}$, $|z|>1$

W.k.T, $z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$= \sum_{n=0}^{\infty} (1)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} = \frac{1}{z^0} + \frac{1}{z^1} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1} = \frac{z}{z-1}$$

2) Find the value of $z\left[(-1)^n\right]$

W.k.T, $z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$= \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n$$

$$= \left(\frac{-1}{z}\right)^0 + \left(\frac{-1}{z}\right)^1 + \left(\frac{-1}{z}\right)^2 + \left(\frac{-1}{z}\right)^3 + \cdots$$

$$= 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots$$

$$= \left(1 + \frac{1}{z}\right)^{-1} = \left(\frac{z+1}{z}\right)^{-1} = \frac{z}{z+1}$$

3. Find the value of $z[a^n]$

$$z[a^n] = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{a^n}{z^n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a}$$

4. PT $z[n] = \frac{z}{(z-1)^2}$

$$z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{0}{z^0} + \frac{1}{z^1} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z^2}\right) + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2} = \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z} \frac{z^2}{(z-1)^2} = \frac{z}{(z-1)^2}$$

5. P.T $z\left[\frac{1}{n+1}\right] = z \log \frac{z}{z-1}$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(n+1) z^n}$$

$$= \frac{1}{0+1\, z^0} + \frac{1}{2 z^1} + \frac{1}{3z^2} + \frac{1}{4z^3} + \cdots$$

$$= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots$$

$$= \frac{z}{z}\left[1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots\right]$$

$$= z\left[\frac{\frac{1}{z}}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right]$$

$$= z \log\left(1 - \frac{1}{z}\right)^{-1} = z \log\left(\frac{z-1}{z}\right)^{-1} = z \log \frac{z}{z-1}$$

6. PT $z\left[\frac{1}{n!}\right] = e^{1/z}$

$$z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n}$$

$$= 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

$$= 1 + \frac{\frac{1}{z}}{1!} + \frac{\frac{1}{z^2}}{2!} + \frac{\frac{1}{z^3}}{3!} + \cdots$$

$$= e^{1/z}$$

Initial Value Theorem:

If $z[f(t)] = F(z)$, then $f(0) = \lim_{z \to \infty} F(z)$

Final Value Theorem:

If $z[f(t)] = F(z)$, then $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$

Problems on IVT & FVT:

1. If $F(z) = \frac{5z}{(z-2)(z-3)}$, Find $f(0)$ and $\lim_{t \to \infty} f(t)$

By IVT,

$$f(0) = \lim_{z \to \infty} F(z) = \lim_{z \to \infty} \frac{5z}{(z-2)(z-3)} = \lim_{z \to \infty} \frac{5z}{z^2-5z+6}$$

$$= \lim_{z \to \infty} \frac{5}{2z-5} \quad \text{[By L'Hôpital's Rule]}$$

$$= 0.$$

By FVT,

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z) = \lim_{z \to 1} (z-1) \frac{5z}{(z-2)(z-3)} = 0.$$

2. Find the Initial & Final value of $F(z) = \frac{z}{2z^2-3z+1}$

By IVT, $\lim_{z \to \infty} F(z) = f(0)$.

$$f(0) = \lim_{z \to \infty} \frac{z}{2z^2-3z+1} = \lim_{z \to \infty} \frac{1}{4z-3}$$

$$= 0.$$

By FVT, $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) \frac{z}{2z^2-3z+1}$$

$$= \lim_{z \to 1} (z-1) \frac{z}{(z-1)(2z-1)}$$

$$= \lim_{z \to 1} \frac{z}{2z-1} = \frac{1}{2(1)-1} = 1$$