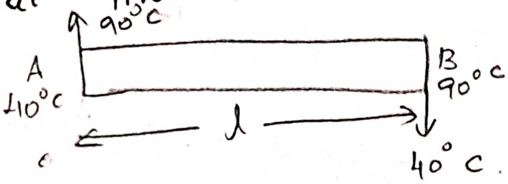
Type - 2 : Steady state conditions with non-zero boundary conditions

1] The ends A and B of a rod of length of 'l' have the temperature 40°c and 90°c until steady state prevails. The temp at A is suddenly raised to 90°c and at the same time that of B is lowered to 40°c. Find the temperature distribution in the rod at time 't'.

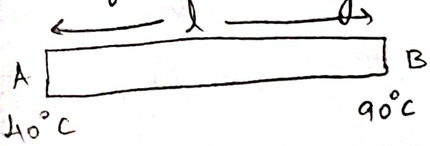


The 1-D HEq is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

First we find the temperature function $u(x,t)$ at any distance before and after the change of temperature at the ends of the rod. So there are two steady state solns.

Before change

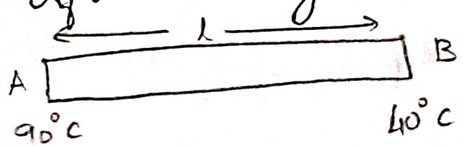


$$u(x) = \frac{b-a}{l} x + a$$

$$= \frac{90-40}{l} x + 40$$

$$= \frac{50x}{l} + 40$$

After change



$$f(x) = \left(\frac{b-a}{l}\right) x + a$$

$$= \left(\frac{40-90}{l}\right) x + 90$$

$$= -\frac{50}{l} x + 90$$

The initial temperature is

$$u(x,0) = \frac{50x}{l} + 40$$

The boundary conditions are

i) $u(0,t) = 90$

ii) $u(l,t) = 40$

iii) $u(x,0) = \frac{50x}{l} + 40$

$\because$ the boundary conditions are non-zero, we assume that

$$u(x,t) = f(x) + v(x,t) \quad -①$$

$$\Rightarrow v(x,t) = u(x,t) - f(x)$$

i) $v(0,t) = u(0,t) - f(0)$

$= 90 - 90 = 0.$

ii) $v(l,t) = u(l,t) - f(l)$

$= 40 - \left(-\frac{50l}{l} + 90\right)$

$= 0.$

iii) $v(x,0) = u(x,0) - f(x)$

$= \frac{50x}{l} + 40 - \left(-\frac{50x}{l} + 90\right)$

$= \frac{100x}{l} - 50.$

The New boundary conditions are

a) $v(0,t) = 0$

b) $v(l,t) = 0$

c) $v(x,0) = \frac{100x}{l} - 50.$

The suitable solution is

$$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-a^2\lambda^2 t} \quad -②$$

apply (a) in ②

$v(0,t)=0.$

$A e^{-a^2\lambda^2 t} = 0.$

$\Rightarrow e^{-a^2\lambda^2 t} \neq 0 \quad \Rightarrow A = 0.$

sub in ②.

$\Rightarrow v(x,t) = B \sin \lambda x \, e^{-a^2\lambda^2 t} \quad — ③$

Apply (b) in ③

$v(l,t)=0$

$B \sin \lambda l \, e^{-a^2\lambda^2 t} = 0.$

$e^{-a^2\lambda^2 t} \neq 0, \quad B \neq 0$ [∵ we get trivial solution]

$\Rightarrow \sin \lambda l = 0$

$\Rightarrow \lambda l = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{l}$

sub in ③.

$v(x,t) = B \sin \frac{n\pi x}{l} \, e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$

The most general solution is

$v(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \, e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \quad — ④$

Apply (c) in ④.

$v(x, 0) = \frac{100x}{l} - 50.$

$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} (1) = \frac{100x}{l} - 50.$

Applying HRSS,

$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$

$= \frac{2}{l} \int_0^l \left(\frac{100x}{l} - 50 \right) \sin \frac{n\pi x}{l} \, dx$

$$u = \frac{100x}{l} - 50 \qquad v = \sin\frac{n\pi x}{l}$$

$$u' = \frac{100}{l} \qquad v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$$

$$u'' = 0 \qquad v_2 = -\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$$

$$B_n = \frac{2}{l}\left[\left(\frac{100x}{l} - 50\right)\left(-\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right) + \frac{100}{l}\,\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[-50\,(-1)^n\frac{l}{n\pi} + (-50)(1)\frac{l}{n\pi}\right]$$

$$= \frac{2}{\cancel{l}}\left[-\frac{50\cancel{l}}{n\pi}\right]\left[1+(-1)^n\right]$$

$$= -\frac{100}{n\pi}\left[1+(-1)^n\right]$$

$$\therefore B_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{-200}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

$$(4) \Rightarrow \quad v(x,t) = \sum_{n=\text{even}}^{\infty} \frac{-200}{n\pi}\sin\frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = f(x) + v(x,t)$$

$$= \frac{-50x}{l} + 90 - \frac{200}{\pi}\sum_{n=\text{even}}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

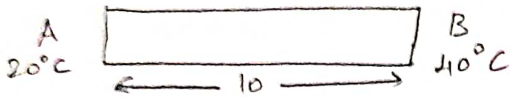
2) A bar 40 cm long with insulated sides, has its ends A and B kept at 20°c and 40°c respectively, until steady state condition prevails. The temperature at A is then suddenly raised to 50°c and at the same instant that of B is lowered to 10°c. Find the subsequent temperature at any point of the bar at any time.

The 1-D Heat eq is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

First we find the temperature function $u(x,t)$ at any distance before and after the changes of temperature at the ends of the rod. So there are two steady state solutions.

Before change

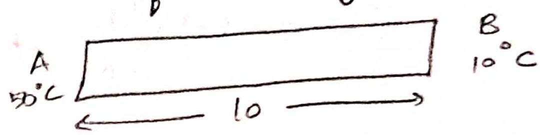


$$u(x) = \left(\frac{b-a}{l}\right)x + a.$$

$$= \left(\frac{40-20}{l}\right)x + 20$$

$$= \frac{20x}{l} + 20.$$

After change



$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{10-50}{l}\right)x + 50$$

$$= \frac{-40x}{l} + 50$$

The initial temperature is $u(x,0) = \frac{20x}{l} + 20.$

The boundary conditions are

i) $u(0,t) = 50$

ii) $u(l,t) = 10.$

iii) $u(x,0) = \frac{20x}{l} + 20.$

$\because$ The boundary conditions are non-zero, we assume that

$$u(x,t) = f(x) + v(x,t) \quad \text{—} ①$$

$$\Rightarrow v(x,t) = u(x,t) - f(x).$$

a) $v(0,t) = u(0,t) - f(0)$

$$= 50 - 50 = 0$$

b) $v(l,t) = u(l,t) - f(l)$

$= 10 - 10 = 0$

c) $v(x,0) = u(x,0) - f(x)$

$= \frac{20x}{l} + 20 - \left(-\frac{40x}{l} + 50\right)$

$= \frac{20x}{l} + 20 + \frac{40x}{l} - 50$

$= \frac{60x}{l} - 30.$

The new boundary conditions are,

a) $v(0,t) = 0$

b) $v(l,t) = 0$

c) $v(x,0) = \frac{60x}{l} - 30.$

The suitable solution is

$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-a^2\lambda^2 t}$ — ②

apply (a) in ②.

$v(0,t) = (A(1) + B(0))e^{-a^2\lambda^2 t}$

$0 = Ae^{-a^2\lambda^2 t}$

$\because e^{-a^2\lambda^2 t} \neq 0 \Rightarrow A = 0.$

sub in ②.

$v(x,t) = B\sin\lambda x\, e^{-a^2\lambda^2 t}$ — ③

apply (b) in ③.

$v(l,t) = B\sin\lambda l\, e^{-a^2\lambda^2 t}$

$0 = B\sin\lambda l\, e^{-a^2\lambda^2 t}$

$\because e^{-a^2\lambda^2 t} \neq 0,\ B \neq 0$ [$\because$ we get trivial soln]

$\sin\lambda l = 0$

$\Rightarrow \lambda l = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{l}$.

sub in (3).

$$V(x,t) = B \sin\frac{n\pi x}{l} e^{-\frac{a^2 n^2 \pi^2 t}{l^2}} \quad -(4)$$

The most general solution is

$$V(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{a^2 n^2 \pi^2 t}{l^2}} \quad -(5)$$

Apply (c) in (5).

$$V(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} (1)$$

$$\frac{60x}{l} - 30 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$u = \frac{60x}{l} - 30$ | $v = \sin\frac{n\pi x}{l}$

$u' = \frac{60}{l}$ | $v_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 0$ | $v_2 = -\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

Apply HRSS,

$$B_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\int_0^l \left(\frac{60x}{l} - 30\right) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{60x}{l} - 30\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \frac{60}{l}\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-30\frac{l}{n\pi}(-1)^n + 0\right] - \left[-30\frac{l}{n\pi}(-1) - 0\right]$$

$$= \frac{2}{l}\left[-30\frac{l}{n\pi}(-1)^n - \frac{30l}{n\pi}\right]$$

$$= \frac{2}{\cancel{l}}\left(\frac{-30\cancel{l}}{n\pi}\right)\left[1 + (-1)^n\right]$$

$$= -\frac{60}{n\pi}\left[1 + (-1)^n\right]$$

$$B_n = \begin{cases} 0, & n \text{ is odd} \\ -\frac{120}{n\pi}, & n \text{ is even} \end{cases}$$

sub in 5

$$v(x,t) = \sum_{n=1}^{\infty} \frac{-120}{n\pi} \sin\frac{n\pi x}{l} \, e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

$$\therefore u(x,t) = f(x) + v(x,t)$$

$$u(x,t) = -\frac{40x}{l} + 50 - \frac{120}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l} \, e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$$

$$= -4x + 50 - \frac{120}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{l} \, e^{-\frac{a^2 n^2 \pi^2 t}{100}}$$