Type - I

Problems based on vibrating string with zero initial velocity

~~1 At string~~ Boundary Conditions.

Given displacement

i) $y(0,t) = 0, \quad t > 0$

ii) $y(l,t) = 0, \quad t > 0$

iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \quad 0 < x < l$

iv) $y(x,0) = f(x)$

Given Velocity.

i) $y(0,t) = 0, \quad t > 0$

ii) $y(l,t) = 0, \quad t > 0$

iii) $y(x,0) = 0$

iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x), \quad 0 < x < l$

1. A string is stretched & fastened to two points $x=0$ & $x=l$ apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of 'x' from one end at a time 't'.

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \text{ where } a^2 = \frac{T}{M}$$

The suitable solution is

$$y(x,t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda at + D \sin \lambda at) \quad \text{— (1)}$$

The boundary conditions are,

i) $y(0,t) = 0, \ t > 0$

ii) $y(l,t) = 0, \ t > 0$ $\Big]$ → boundary condition

iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \ 0 < x < l$

iv) $y(x,0) = f(x), \ 0 < x < l$ $\Big]$ → Initial condition.

$= k(lx - x^2), \ 0 < x < l.$

Applying condition (i) in (1)

$$y(0,t) = (A \cos \lambda(0) + B \sin \lambda(0))(C \cos \lambda at + D \sin \lambda at)$$

$$0 = A(C \cos \lambda at + D \sin \lambda at)$$

$A = 0, \ C \cos \lambda at + D \sin \lambda at \neq 0.$

Applying $A = 0$ in (1).

$$y(x,t) = B \sin \lambda x \,[C \cos \lambda at + D \sin \lambda at] \quad \text{— (2)}$$

Applying condition (ii) in (2).

$$y(l,t) = B \sin \lambda l \,[C \cos \lambda at + D \sin \lambda at]$$

$$0 = B \sin \lambda l \,[C \cos \lambda at + D \sin \lambda at]$$

$B \neq 0, \ \sin \lambda l = 0, \ C \cos \lambda at + D \sin \lambda at \neq 0, \ \forall t$

[∵ If $B = 0$, then we get a trivial solution]

$\therefore \sin \lambda l = 0.$

$\lambda l = n\pi$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

Applying λ in (2)

$$y(x,t) = B \sin \frac{n\pi x}{l} \left[C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right] \quad -(3)$$

Applying condition (iii) in (3)

diff p.w.r.t 't'

$$\frac{\partial y}{\partial t}(x,t) = B \sin \frac{n\pi x}{l} \left[-C \sin \frac{n\pi at}{l} \left(\frac{n\pi a}{l}\right) + D \cos \frac{n\pi at}{l} \left(\frac{n\pi a}{l}\right) \right]$$

when $t = 0$.

$$\frac{\partial y}{\partial t}(x,0) = B \sin \frac{n\pi x}{l} \left[-C(0) + D \cos(0) \left(\frac{n\pi a}{l}\right) \right]$$

$$0 = \left[B \sin \frac{n\pi x}{l} \right] \left[D \frac{n\pi a}{l} \right]$$

$B \sin \frac{n\pi x}{l} \neq 0$, $D = 0$, $\frac{n\pi a}{l} \neq 0$.

Applying $D = 0$ in (3)

$$y(x,t) = B \sin \frac{n\pi x}{l} \cdot C \cos \frac{n\pi at}{l}$$

$$= BC \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

Take $BC = b_n$.

$$y(x,t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad -(4)$$

Applying condition (iv) in (4).

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a(0)}{l}$$

$$K(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Expand b_n in half range series,

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l K(lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l}\, dx.$$

By applying Bernoulli's formula,

$u = lx - x^2$

$u' = l - 2x$

$u'' = -2$

$V = \sin\frac{n\pi x}{l}$

$V_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$V_2 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$V_3 = \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}$

$$b_n = \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) - 2\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$= \frac{2k}{l}\left\{\left[0 + 0 - 2\frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)^3}\right] - \left[0 + 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]\right\}$$

$$= \frac{2k}{l}\left[-\frac{2(-1)^n l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right]$$

$$b_n = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$\therefore b_n = \begin{cases} \frac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Applying b_n in (4)

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=\text{odd}}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

2. A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, then find the displacement.

The 1-D wave eq is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are,

i) $y(0,t) = 0, \quad \forall t$

ii) $y(l,t) = 0, \quad \forall t$

iii) $\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0, \forall x$

iv) $y(x,0) = f(x) = y_0 \sin^3 \frac{\pi x}{l}$

The suitable solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad -①$$

Applying (i) in ① we get

$y(0,t) = 0.$

$[A(1) + B(0)][C\cos\lambda at + D\sin\lambda at] = 0.$

$A[C\cos\lambda at + D\sin\lambda at] = 0.$

$C\cos\lambda at + D\sin\lambda at \neq 0 \Rightarrow A = 0.$

sub in ①

$$y(x,t) = B\sin\lambda x[C\cos\lambda at + D\sin\lambda at] \quad -②$$

Applying (ii) in ② we get

$y(l,t) = 0$

$B\sin\lambda l[C\cos\lambda at + D\sin\lambda at] = 0,$

$B \neq 0, \quad C\cos\lambda at + D\sin\lambda at \neq 0;$

$\Rightarrow \sin \lambda l = 0.$

$\Rightarrow \lambda l = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{l}.$

sub λ in ②.

$$y(x,t) = B \sin \frac{n\pi x}{l}\left[C \cos \frac{n\pi a t}{l} + D \sin \frac{n\pi a t}{l}\right] \quad - ③$$

Applying (iii) in ③

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0.$$

$$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l}\left[-C \sin \frac{n\pi a t}{l}\left(\frac{n\pi a}{l}\right) + D \cos \frac{n\pi a t}{l}\left(\frac{n\pi a}{l}\right)\right]$$

put $t = 0$

$$0 = B \sin \frac{n\pi x}{l}\left[0 + D\left(\frac{n\pi a}{l}\right)\right]$$

$B \neq 0, \quad \sin \frac{n\pi x}{l} \neq 0 \quad \Rightarrow D = 0.$

sub $D=0$ in ③.

$$y(x,t) = B \sin \frac{n\pi x}{l}\, C \cos \frac{n\pi a t}{l}$$

$$= BC \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

$$= b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \quad - ④$$

Applying condition (iv) in ④ we get

$$y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$$

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = y_0 \sin^3 \frac{\pi x}{l}$$

$$= y_0\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right]$$

$$\Rightarrow b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \cdots = \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}$$

Equating the like co-efficients

$$b_1 = \frac{3y_0}{4}, \quad b_2 = 0, \quad b_3 = -\frac{y_0}{4}, \quad b_4 = b_5 = \cdots = 0.$$

Substitute the above values in (4)

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}$$