Method of Residues:

$f(n) =$ Sum of residues of $z^{n-1} f(z)$ at its poles.

Note:

1. The pole of order 1 at $z = a$ is

$$\{\text{Res } z^{n-1} f(z)\}_{z=a} = \lim_{z \to a} (z-a) \left[z^{n-1} f(z)\right].$$

2. The pole of order m at $z = a$ is

$$\{\text{Res } z^{n-1} f(z)\}_{z=a} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m z^{n-1} f(z)$$

Problem on Cauchy's Residue Method:

1. Find $z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

Let $F(z) = \frac{z}{(z-1)(z-2)}$

$$z^{n-1} f(z) = z^{n-1} \frac{z}{(z-1)(z-2)} = \frac{z^n}{(z-1)(z-2)}$$

Here $z=1$ is a pole of order 1

$z=2$ is a pole of order 1

$$\text{Res}\{z^{n-1} f(z)\}_{z=1} = \lim_{z \to 1} (z-1) \frac{z^n}{(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{z^n}{z-2}.$$

$$= -1$$

$$\text{Res}\{z^{n-1} f(z)\}_{z=2} = \lim_{z \to 2} (z-2) \frac{z^n}{(z-1)(z-2)}$$

$$= \lim_{z \to 2} \frac{z^n}{z-1}$$

$$= 2^n$$

$\therefore f(n) =$ sum of residues of $z^{n-1} f(z)$ at its pole

$$= -1 + 2^n$$

2. Find $z^{-1}\left[\frac{z(z+1)}{(z-1)^2}\right]$

$$f(z) = \frac{z(z+1)}{(z-1)^2}$$

$$z^{n-1} f(z) = z^{n-1} \frac{z(z+1)}{(z-1)^2}$$

$$= \frac{z^n (z+1)}{(z-1)^2}$$

Here $z=1$ is a pole of order 2.

$$\text{Res}\left\{z^{n-1} f(z)\right\}_{z=1} = \lim_{z\to 1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \cdot \left[(z-1)^2 \frac{z^n (z+1)}{(z-1)^2}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[z^n (z+1)\right]$$

$$= \lim_{z\to 1}\left[z^n (1) + (z+1) n z^{n-1}\right]$$

$$= 1 + 2n$$

$\therefore f(n) =$ sum of residues $= 1 + 2n$.

3. Find $z^{-1}\left[\frac{z^2}{z^2+4}\right]$ using Cauchy's Residue method.

Let $f(z) = \frac{z^2}{z^2+4}$

$$z^{n-1} f(z) = z^{n-1} \frac{z^2}{z^2+4} = \frac{z^{n+1}}{z^2+4} = \frac{z^{n+1}}{(z+2i)(z-2i)}$$

$z = 2i$ is a pole of order 1

$z = -2i$ is a pole of order 1.

$$\text{Res}\left[z^{n-1} f(z)\right] = \lim_{z\to 2i} (z-2i) \frac{z^{n+1}}{z^2+4}$$

$$= \lim_{z\to 2i} (z-2i) \frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$= \frac{(2i)^{n+1}}{2i+2i} = \frac{(2i)^n \cdot 2i}{2(2i)} = \frac{2^n i^n}{2} = 2^{n-1} i^n$$

$$\text{Res}\left[z^{n-1} f(z)\right] = \lim_{z \to -2i} (z+2i) \frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$= \lim_{z \to -2i} \frac{z^{n+1}}{z-2i}$$

$$= \frac{(-2i)^{n+1}}{-4i}$$

$$= \frac{(-2i)^n \cdot (-2i)'}{2(-2i)}$$

$$= \frac{2^n (-i)^n}{2}$$

$$= 2^{n-1} (-i)^n$$

$\therefore f(n) =$ sum of the residues

$$= 2^{n-1} (i)^n + (2^{n-1}) (-i)^n$$

$$= 2^{n-1} \left[\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} + \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}\right]$$

$$= 2^{n-1} \left[2 \cos \frac{n\pi}{2}\right]$$

$$= 2^n \cos \frac{n\pi}{2}$$

HW. 1) $z^{-1} \left[\frac{z}{(z-1)(z^2+1)}\right]$