Type - II

Problems based on vibrating string with non-zero initial velocity:

1) A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$. Find the displacement.

The 1-D wave equation,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

i) $y(0,t)=0, \; \forall t$

ii) $y(l,t)=0, \; \forall t$

iii) $y(x,0)=0, \; \forall x$

iv) $\left.\frac{\partial y}{\partial t}\right]_{t=0} = f(x)$

The suitable solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$$ ──(1)

Applying condition (i) in ① we get

$y(0,t)=0.$

$(A(1)+B(0))(c\cos\lambda at + D\sin\lambda at)=0.$

$A(c\cos\lambda at + D\sin\lambda at)=0$

$c\cos\lambda at + D\sin\lambda at \neq 0 \Rightarrow A=0.$

sub in ① $\Rightarrow y(x,t) = B\sin\lambda x\,(c\cos\lambda at + D\sin\lambda at)$ — ②

Applying condition (ii) in ② we get

$y(l,t)=0.$

$B\sin\lambda l\,(c\cos\lambda at + D\sin\lambda at)=0.$

$c\cos\lambda at + D\sin\lambda at \neq 0,\ B\neq 0$ [then we get trivial soln]

$\therefore \sin\lambda l = 0 \Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}$

sub in ②

$y(x,t) = B\sin\frac{n\pi x}{l}\left[c\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right]$ — ③

Applying condition (iii) in ③

$y(x,0)=0.$

$B\sin\frac{n\pi x}{l}\,[c(1)+D(0)] = 0.$

$BC\sin\frac{n\pi x}{l} = 0.$

$B\neq 0,\ \sin\frac{n\pi x}{l}\neq 0 \Rightarrow \boxed{C=0}$

sub in ③

$y(x,t) = B\sin\frac{n\pi x}{l}\ D\sin\frac{n\pi at}{l}$

$$= BD \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

Let $BD = B_n$.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \quad \text{---} (4)$$

Applying condition (iv) in (4).

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = f(x)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \left(\frac{n\pi a}{l}\right)$$

put $t=0$

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} (1) \frac{n\pi a}{l}$$

$$\lambda(lx - x^2) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\left(\frac{n\pi a}{l}\right)$$

Put $B_n\left(\frac{n\pi a}{l}\right) = b_n$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \lambda(lx - x^2)$$

Applying Half range sine series,

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx.$$

$$= \frac{2}{l}\int_0^l \lambda(lx - x^2) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l}\, dx.$$

$$= \frac{2\lambda}{l}\left[(lx - x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right] + (l-2x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} - 2\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$= \frac{2\lambda}{l}\left[0 + 0 - 2\frac{(-1)^n l^3}{n^3\pi^3}\right] - \left[0 + 0 - \frac{2(1)l^3}{n^3\pi^3}\right]$$

$$b_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right].$$

$$B_n \frac{n\pi a}{l} = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{l}{n\pi a}\,\frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4\lambda l^3}{n^4\pi^4 a}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} \frac{8\lambda l^3}{n^4\pi^4 a} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}$$

$$\therefore (4) \Rightarrow y(x,t) = \sum_{n=odd}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{n^4\pi^4 a} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{\pi^4 a} \sum_{n=odd}^{\infty} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

2. If a string of length 'l' is initially at rest in its equilibrium position and each of its points is given by the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = V_0 \sin^3\frac{\pi x}{l}$, Determine the displacement function $y(x,t)$, $0 < x < l$

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are,

i) $y(0,t) = 0$, $\forall t$

ii) $y(l,t) = 0$, $\forall t$

iii) $y(x,0) = 0$.

iv) $\left.\frac{\partial y}{\partial t}\right]_{t=0} = V_0 \sin^3\frac{\pi x}{l}$

The suitable solution is

$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$ — ①

Applying (i) in ① we get

$y(0,t) = 0.$

$(A(1) + B(0))(C\cos\lambda at + D\sin\lambda at) = 0$

Here $C\cos\lambda at + D\sin\lambda at \neq 0.$

$\Rightarrow A = 0.$

① $\Rightarrow y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at)$ — ②

Applying (ii) in ② we get,

$y(l,t) = 0.$

$B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$

$B \neq 0,\ C\cos\lambda at + D\sin\lambda at \neq 0.$

$\Rightarrow \sin\lambda l = 0 \quad \Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}$

sub in ②,

$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right]$ — ③

Applying (iii) in ③,

$y(x,0) = 0.$

$B\sin\frac{n\pi x}{l}\left[C(1) + D(0)\right] = 0.$

$BC\sin\frac{n\pi x}{l} = 0.$

$B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0 \quad \Rightarrow C = 0.$

sub in ③.

$y(x,t) = B\sin\frac{n\pi x}{l}\ D\sin\frac{n\pi at}{l}$

$= BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

$B_n = BD$ (say)

$$y(x,t) = B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

The most general soln is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l} \quad - (4)$$

~~Before~~ applying (iv) in (4),

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}\left(\frac{n\pi a}{l}\right)$$

Put $t=0$.

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{l}\right) \sin\frac{n\pi x}{l}$$

$$V_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{l}\right) \sin\frac{n\pi x}{l}$$

$$V_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right] = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{l}\right) \sin\frac{n\pi x}{l}$$

$$\frac{3V_0}{4}\sin\frac{\pi x}{l} - \frac{V_0}{4}\sin\frac{3\pi x}{l} = B_1\frac{\pi a}{l}\sin\frac{\pi x}{l} + B_2\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + B_3\frac{3\pi a}{l}\sin\frac{3\pi x}{l}$$

Equating the like co-efficients we get,

$$B_1\frac{\pi a}{l} = \frac{3V_0}{4}, \quad B_2\frac{2\pi a}{l} = 0, \quad B_3\frac{3\pi a}{l} = -\frac{V_0}{4}, \quad B_4 = B_5 = \dots = 0$$

$$\therefore B_1 = \frac{3V_0 l}{4\pi a}, \quad B_2 = 0, \quad B_3 = \frac{-V_0 l}{12\pi a}, \quad B_4 = B_5 = \dots = 0$$

sub in (4), we get

$$y(x,t) = \frac{3V_0 l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi a t}{l} - \frac{V_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi a t}{l}$$