Properties

1. Linear property:

$$z[af(n)+bg(n)] = aF(z)+bG(z), \quad z[f(n)] = F(z), \quad z[g(n)] = G(z)$$

'a' and 'b' are constants.

Proof:

$$z[af(n)+bg(n)] = \sum_{n=0}^{\infty} [af(n)+bg(n)]z^{-n}$$

$$= \sum_{n=0}^{\infty} af(n)z^{-n} + \sum_{n=0}^{\infty} bg(n)z^{-n}$$

$$= a\, z[f(n)] + b\, z[g(n)]$$

$$= F(z)+G(z).$$

2. First shifting property:

If $z[f(t)] = F(z)$, then $z[e^{-at} f(t)] = F[z e^{aT}]$

(or)

$$z[e^{-at} f(t)] = \{F(z)\}_{z \to ze^{aT}}, \quad z[e^{at} f(t)] = \{F(z)\}_{z \to ze^{-aT}}$$

Proof:

$$\because z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \cdots$$

$$z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (z e^{aT})^{-n}$$

$$= F[z e^{aT}]$$

3) Change of scale:

If $z[f(n)] = F(z)$ then $z[a^n f(n)] = F\left(\frac{z}{a}\right)$.

Proof:

$$z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\Rightarrow z[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n) z^{-n} = \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right) = \{F(z)\}_{z \to \frac{z}{a}}$$

4) Second shifting property:

If $z[f(n)] = F(z)$ then $z[f(n+1)] = zF(z) - z f(0)$

5) Differentiation in z-domain:

i) $z[n f(n)] = -z \frac{d}{dz}\{F(z)\}$, where $F(z) = z[f(n)]$

ii) $z[n f(t)] = -z \frac{d}{dz}\{F(z)\}$, where $F(z) = z[f(t)]$

Proof:

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\frac{d}{dz} F(z) = \frac{d}{dz} \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} f(n) (-n) z^{-n-1} = \frac{-n}{z} \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= \frac{-1}{z} \sum_{n=0}^{\infty} n f(n) z^{-n} = \frac{-1}{z} z[n f(n)]$$

$$z[n f(n)] = -z \frac{d}{dz} F(z)$$

1. Find $z[e^{-iat}]$

$$z[e^{-iat}] = z[e^{-iat}(1)] = [z(1)]_{z \to z e^{iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to z e^{iaT}}$$

$$= \frac{z e^{iaT}}{z e^{iaT} - 1}$$

2. Find $z[\cos at]$ and $z[\sin at]$

$$z[e^{iat}] = z[e^{iat}(1)] = [z(1)]_{z \to z e^{-iaT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to z e^{-iaT}}$$

$$= \frac{z e^{-iaT}}{z e^{-iaT} - 1}$$

$\div$ by e^{-iaT}

$$= \frac{z}{z - e^{iaT}}$$

$$z[\cos at + i \sin at] = \frac{z}{z - (\cos aT + i \sin aT)}$$

$$= \frac{z}{(z - \cos aT) - i \sin aT} \times \frac{(z - \cos aT) + i \sin aT}{(z - \cos aT) + i \sin aT}$$

$$= \frac{z(z - \cos aT) + iz \sin aT}{(z - \cos aT)^2 + (\sin aT)^2}$$

$$z[\cos at] + iz[\sin at] = \frac{z(z - \cancel{a} \cos aT) + iz \sin aT}{z^2 - 2z \cos aT + 1}$$

Equating the real and imaginary parts we have

$$z[\cos at] = \frac{z(z - \cos aT)}{z^2 - 2z\cos aT + 1}, \quad z[\sin aT] = \frac{z \sin aT}{z^2 - 2z\cos aT + 1}$$