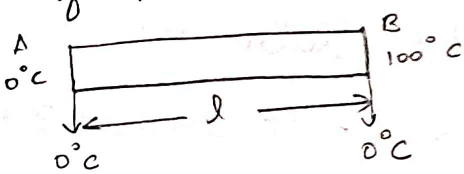
Steady State condition:

The state in which the temperature depends only on the distance but not on time 't' is called steady state. $\therefore u(x,t)$ becomes $u(x)$ under steady state.

Note: $u(x) = \left(\frac{b-a}{l}\right)x + a$.

Type-1 - Steady state conditions both ends at zero temperature

The ends A and B of a rod of length 'l' have the temperature $0°c$ and $100°c$ until steady state condition prevail. If the temperature at B is reduced suddenly to $0°c$ and kept so while that of A is maintained, find the temperature $u(x,t)$ at a distance 'x' from A and at a time 't'



The 1-D HE is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are,

i) $u(0,t)=0$

ii) $u(l,t)=0$

iii) $u(x,0) = f(x) = \frac{100x}{l}$

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$
$$= \left(\frac{100-0}{l}\right)x + 0$$
$$= \frac{100x}{l}$$

The suitable soln is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-a^2\lambda^2 t} \quad -①$$

Applying (i) in ①, we get

$$u(0,t) = [A(1) + B(0)]\, e^{-a^2\lambda^2 t}$$

$$0 = A e^{-a^2\lambda^2 t}$$

Here $e^{-a^2\lambda^2 t} \neq 0 \Rightarrow A = 0$.

sub in ①.

$$\therefore u(x,t) = B\sin\lambda x\, e^{-a^2\lambda^2 t} \quad -②$$

Applying (ii) in ② we get

$u(l,t) = B \sin\lambda l \, e^{-a^2\lambda^2 t}$

$0 = B \sin\lambda l \, e^{-a^2\lambda^2 t}$

Here $e^{-a^2\lambda^2 t} \neq 0$, $B \neq 0$ [$\because$ we get trivial soln]

$\Rightarrow \sin\lambda l = 0$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}$

sub in ② we get

$$u(x,t) = B \sin\frac{n\pi x}{l} \, e^{-a^2 \frac{n^2\pi^2}{l^2} t} \quad -③$$

The most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \, e^{-a^2 \frac{n^2\pi^2}{l^2} t} \quad -④$$

Applying (iii) in ④ we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \, (1)$$

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad [\because B_n = b_n]$$

Apply Half range Sine Series,

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$u = x$ $\quad V = \sin\frac{n\pi x}{l}$

$u' = 1$ $\quad V_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 0$ $\quad V_2 = -\frac{\sin\frac{n\pi x}{l}}{(\frac{n\pi}{l})^2}$

$$b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{200}{l^2}\int_0^l x \sin\frac{n\pi x}{l}\,dx.$$

$$b_n = \frac{200}{l^2}\left[-x\frac{\cos\frac{n\pi x}{l}}{(\frac{n\pi}{l})} + 1\frac{\sin\frac{n\pi x}{l}}{(\frac{n\pi}{l})^2}\right]_0^l$$

$$= \frac{200}{l^2}\left[-l\frac{\cos n\pi}{(\frac{n\pi}{l})} + 0\right] - \left[0 + 0\right]$$

$$= \frac{200}{l^2}\left[-l^2 \frac{(-1)^n}{n\pi}\right]$$

$$b_n = \frac{-200(-1)^n}{n\pi}$$

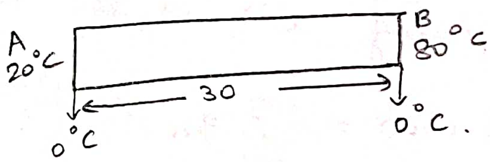
$$\therefore B_n = \frac{200(-1)^{n+1}}{n\pi}$$

sub in (4) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{l} e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

2. A rod of 30cm long has its ends A & B kept at 20° and 80°c respectively until steady state condition prevails. The temperature at each end is then suddenly reduced to 0°c and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.



The 1-D HE is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

i) $u(0,t)=0$

ii) $u(l,t)=0 \Rightarrow u(30,t)=0$

iii) $u(x,0) = f(x) = 2x+20$.

$$f(x) = \left(\frac{b-a}{l}\right)x + a$$

$$= \left(\frac{80-20}{30}\right)x + 20$$

$$= \frac{60}{30}x + 20$$

$$= 2x + 20$$

The suitable soln is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-a^2\lambda^2 t} \quad \text{—(1)}$$

Applying (i) in (1)

$$u(0,t) = [A(1) + B(0)]e^{-a^2\lambda^2 t}$$

$$0 = Ae^{-a^2\lambda^2 t}$$

$e^{-a^2\lambda^2 t} \neq 0 \;.\; \Rightarrow A = 0$

sub in (1)

$u(x,t) = B \sin \lambda x \, e^{-a^2\lambda^2 t}$ — (2)

Applying (ii) in (2).

$u(30,t) = B \sin \lambda (30) \, e^{-a^2\lambda^2 t}$

$0 = B \sin 30\lambda \, e^{-a^2\lambda^2 t}$

$e^{-a^2\lambda^2 t} \neq 0$, $B \neq 0$ [$\because$ we get trivial soln]

$\therefore \sin 30\lambda = 0$

$\Rightarrow 30\lambda = n\pi$

$\lambda = \frac{n\pi}{30}$

sub in (2) we get

$u(x,t) = B \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2\pi^2}{900} t}$ — (3)

The most general soln is

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{-a^2 \frac{n^2\pi^2}{900} t}$ — (4)

Applying (iii) in (4) we get

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} (1)$

$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30}$ [$\because B_n = b_n$]

$2x+20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30}$

Applying Half range Sine Series,

$u = 2x+20$ | $v = \sin \frac{n\pi x}{30}$

$u' = 2$ | $v_1 = -\frac{\cos \frac{n\pi x}{30}}{\frac{n\pi}{30}}$

$u'' = 0$ | $v_2 = -\frac{\sin \frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)^2}$

$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$b_n = \frac{2}{30} \int_0^{30} (2x+20) \sin \frac{n\pi x}{30} dx$.

$\therefore b_n = \frac{2}{30} \left[-(2x+20) \frac{\cos \frac{n\pi x}{30}}{\frac{n\pi}{30}} + 2 \frac{\sin \frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)^2} \right]_0^{30}$

$= \frac{1}{15} \left[-80(30) \frac{\cos n\pi}{n\pi} + 2 \frac{(900)}{n\pi}(0) \right] - \left[20 \frac{(1)}{n\pi}(30) + 0 \right]$

$= \frac{1}{15} \left[-2400 \frac{(-1)^n}{n\pi} + \frac{600}{n\pi} \right]$

$$b_n = \frac{1}{15}\,\frac{600}{n\pi}\left[1 - 4(-1)^n\right]$$

$$\therefore B_n = \frac{40}{n\pi}\left[1 - 4(-1)^n\right]$$

sub in (4) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[1 - 4(-1)^n\right] \sin\frac{n\pi x}{30}\, e^{-\frac{a^2 n^2 \pi^2}{900}t}$$

$$\therefore u(x,t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[1 - 4(-1)^n\right] \sin\frac{n\pi x}{30}\, e^{-\frac{a^2 n^2 \pi^2}{900}t}$$