Therefore, strain = □ L t / L

= □ t

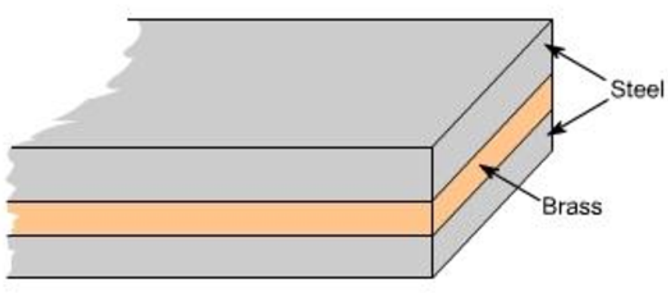
Therefore ,the stress generated in the material by the application of sufficient force to remove this strain

= strain x E

or Stress = E □ t

Consider now a compound bar constructed from two different materials rigidly joined together, for simplicity.

Let us consider that the materials in this case are steel and brass.



If we have both applied stresses and a temp. change, thermal strains may be added to those given by generalized hook's law equation –e.g.

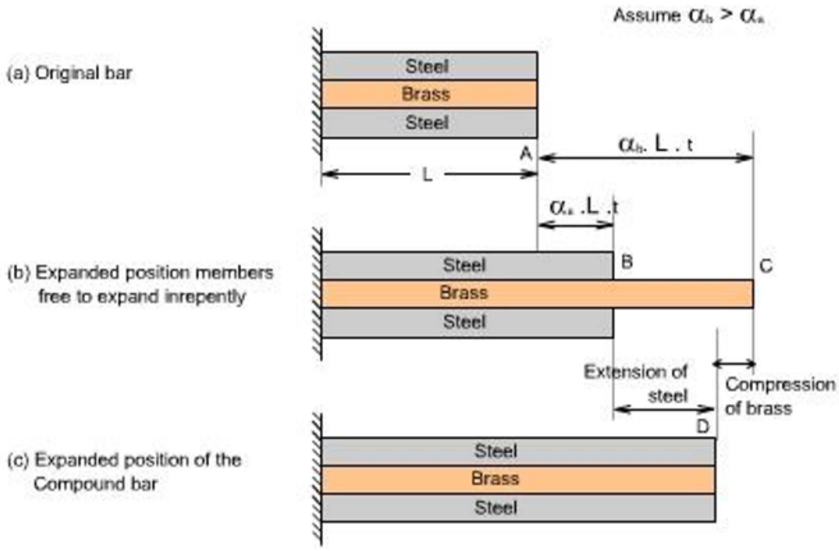
$$\in_x = \frac{1}{E}\left[\sigma_x - \gamma(\sigma_y + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_y - \gamma(\sigma_x + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_z - \gamma(\sigma_x + \sigma_y)\right] + \alpha\Delta t$$

While the normal strains a body are affected by changes in temperatures, shear strains are not. Because if the temp. of any block or element changes, then its size changes not its shape therefore shear strains do not change.

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temp. rises each material will attempt to expand by different amounts. Figure below shows the positions to which the individual materials will expand if they are completely free to expand (i.e not joined rigidly together as a compound bar). The extension of any Length L is given by □ L t



In general, changes in lengths due to thermal strains may be calculated form equation $\square_t$ = □ Lt, provided that the members are able to expand or contract freely, a situation that exists in statically determinates structures. As a consequence no stresses are generated in a statically determinate structure when one or more members undergo a uniform temperature change. If in a structure (or a compound bar), the free expansion or contraction is not allowed then the member becomes s statically indeterminate, which is just being discussed as an example of the compound bar and thermal stresses would be generated.

Since in this case the coefficient of expansion of the brass $\square_B$ is greater then that for the steel $\square_s$. the initial lengths L of the two materials are assumed equal.

If the two materials are now rigidly joined as a compound bar and subjected to the same temp. rise, each materials will attempt to expand to its free length position but each will be affected by the movement of the other. The higher coefficient of expansion material (brass) will therefore, seek to pull the steel up to its free length position and conversely, the lower coefficient of expansion martial (steel) will try to hold the brass back. In practice a compromised is reached, the compound bar extending to the position shown in fig (c), resulting in an effective compression of the brass from its free length position and an effective extension of steel from its free length position.

Therefore, from the diagrams,we may conclude thefollowing

Conclusion 1.

Extension of steel + compression brass = difference in “ free” length

Applying Newton 's law of equal action and reaction the following second Conclusion also holds good.

Conclusion 2.

The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to long member by the short member.

Thus in this case

Tensile force in steel = compressive force in brass

These conclusions may be written in the form of mathematical equations as given below:

for conclusion 1

$$\frac{\sigma_s . L}{E_s} + \frac{\sigma_B . L}{E_B} = (\alpha_B - \alpha_s) L . t$$

for conclusion 2

$$\sigma_s . A_s = \sigma_B . A_B$$

Using these two equations, the magnitude of the stresses may be determined.

Strain Energy

Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy** :

Strain Energy in uniaxial Loading

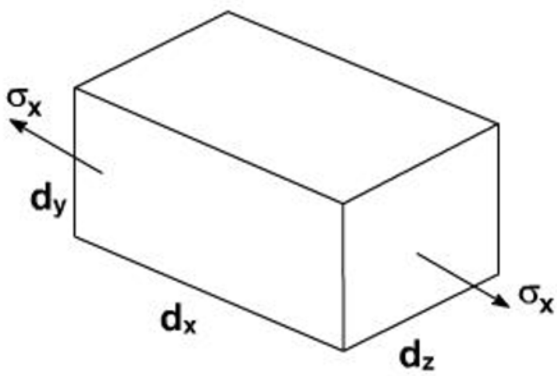


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress σ_x.

The forces acting on the face of this element is σ_x. dy. dz

where

dydz = Area of the element due to the application of forces, the element deforms to an amount = σ_x dx

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain as shown in Fig . 2.

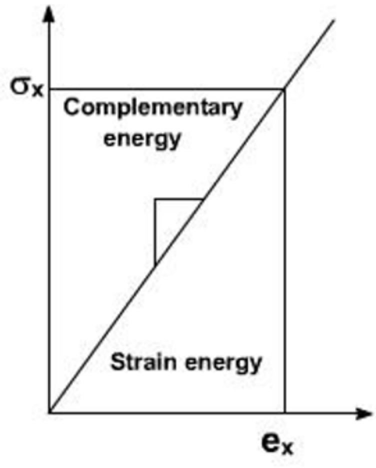


Fig .2

From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

For a perfectly elastic body the above work done is the internal strain energy "du".

$$\boxed{du = \frac{1}{2}\sigma_x dydz\, \epsilon_x\, dx} \qquad .....(2)$$

$$= \frac{1}{2}\sigma_x \epsilon_x\, dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x \epsilon_x\, dv} \qquad ....(3)$$

where dv = dxdydz

= Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x \epsilon_x} \qquad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy – density 'u_o' .

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^2}{2E} = \frac{E\epsilon_x^2}{2}} \qquad ......(5)$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^2}{2E} dv} \qquad ....(6)$$

In the case of a rod of uniform cross – section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

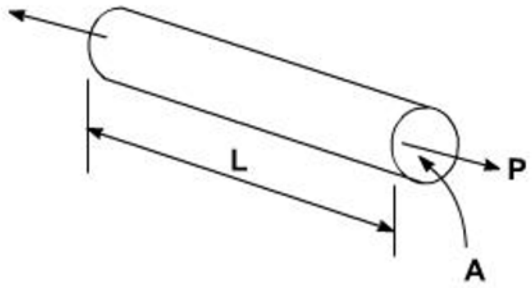


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^2}{2E} dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_{0}^{L} \frac{P^2}{2EA^2} A dx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.

L = Length of the bar

$$\boxed{U = \frac{P^2 L}{2AE}} \qquad .....(7)$$

Modulus of resilience :

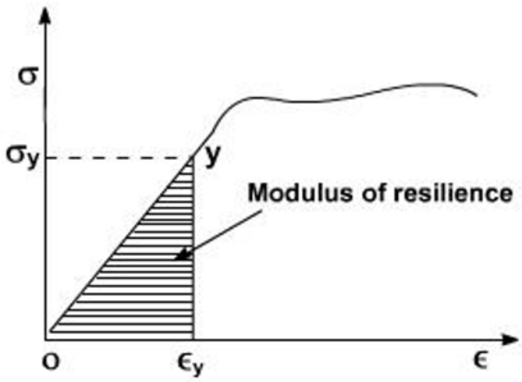


Fig .4

Suppose ' σ_x' in strain energy equation is put equal to σ_y i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

So $$\boxed{U_y = \frac{\sigma_y^2}{2E}} \qquad .....(8)$$

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion 'OY' of the stress – strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

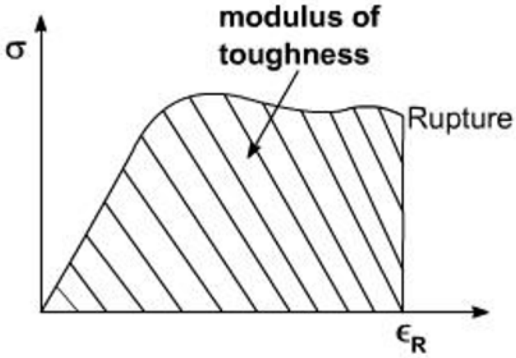


Fig .5

Suppose 'ϵ' [strain] in strain energy expression is replaced by ϵ_R strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\,\epsilon_x\,dx = \frac{E\,\epsilon_R^2}{2}\,dv$$

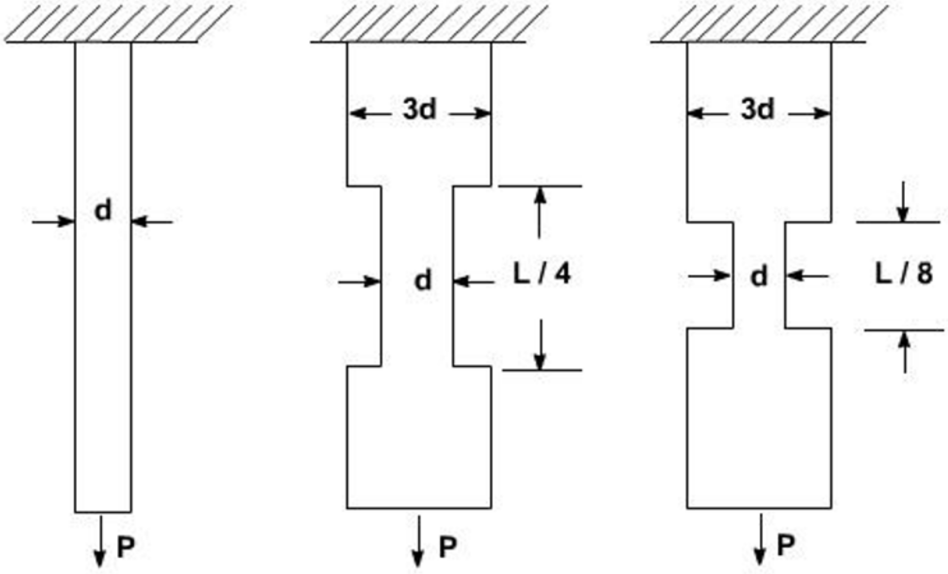
$$\boxed{U = \frac{E\,\epsilon_R^2}{2}} \qquad \ldots\ldots(9)$$

From the stress – strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length 'L' but different shapes are shown in fig below. The first bar has a diameter 'd' over its entire length, the second had this diameter over one – fourth of its length, and the third has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2(L/4)}{2EA} + \frac{P^2(3L/4)}{2E9A} = \frac{P^2L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

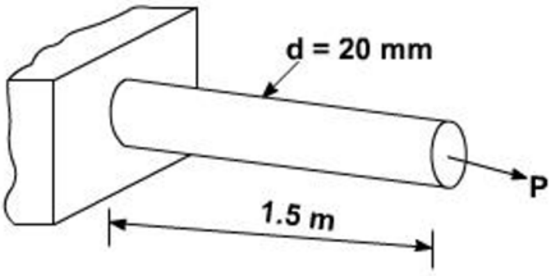
$$U_3 = \frac{P^2(L/8)}{2EA} + \frac{P^2(7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution :

Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4} d^2 L$$

$$= \frac{\pi}{4} 20 \times 1.5 \times 10^3$$

$$= 471 \times 10^3 \ mm^3$$

Yield Strength :

As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to σ_x .

$$U = \frac{\sigma_y^2}{2E}$$

$$0.144 = \frac{\sigma_y^2}{2 \times (200 \times 10^3)}$$

$$\boxed{\sigma_y = 200 \text{ Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

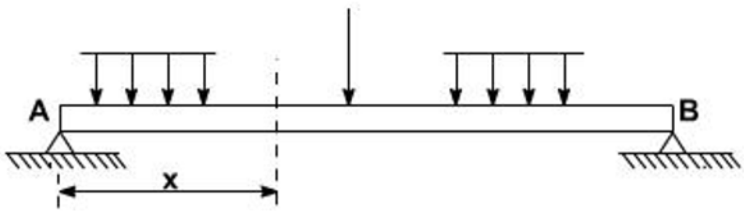


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\sigma = \frac{MY}{I}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$= \int \frac{M^2 . y^2}{2EI^2} dv$ .....(10)

Substituting dv = dxdA

Where dA = elemental cross-sectional area

$\frac{M^2 . y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

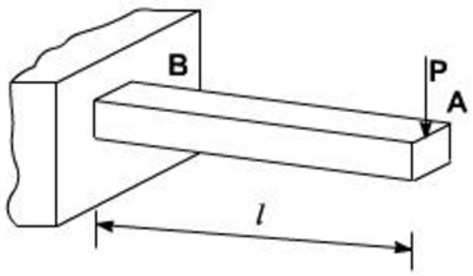
$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx$ .....(11)

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$U = \int_0^L \frac{M^2}{2EI} dx \quad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.



Solution : The bending moment at a distance x from end A is defined as

$$M = -Px$$

Substituting the above value of M in the expression of strain energy we may write

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$U = \int_0^L \frac{P^2 L^3}{EI}$$

Problem 2 :

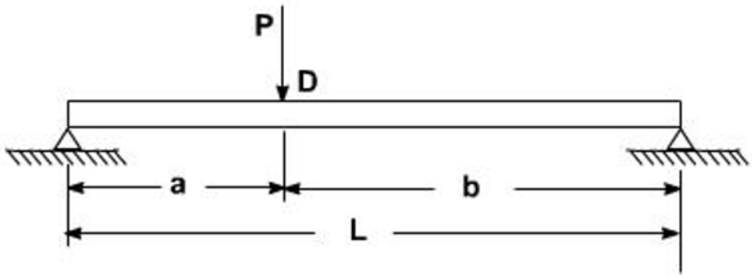
a. Determine the expression for strain energy of the prismatic beam AB for the loading as sh in figure below. Take into account only the effect of normal stresses due to bendin

b. Evaluate the strain energy for the following values of the beam

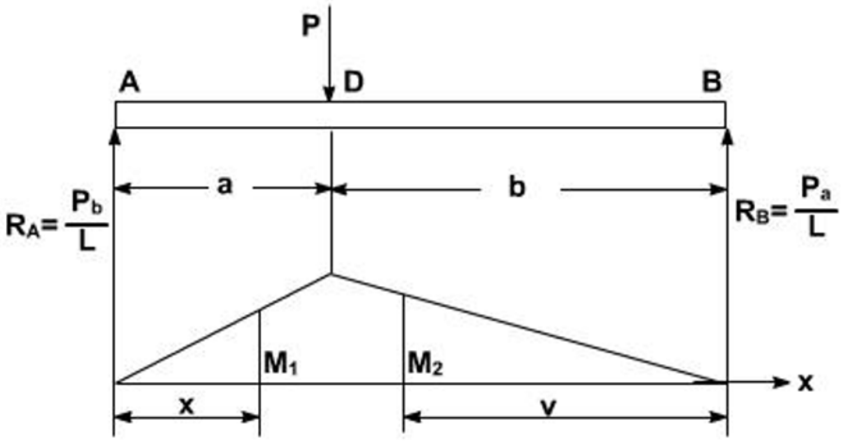
P = 208 KN ; L = 3.6 m = 3600 mm

A = 0.9 m = 90mm ; b = 2.7m = 2700 mm

E = 200 GPa ; I = 104 x 10^8 mm^4



Solution:

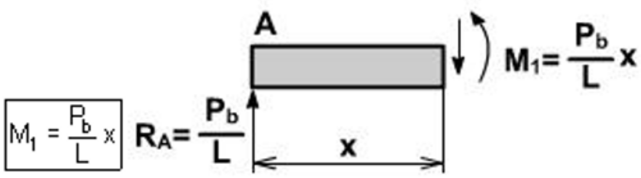


a.

Bending Moment : Using the free – body diagram of the entire beam, we may determine the values of reactions as follows:

$R_A = P_b / L$ $R_B = P_a / L$

For Portion AD of the beam, the bending moment is



For Portion DB, the bending moment at a distance v from end B is

$$M_2 = \frac{P_a}{L} x$$

Strain Energy :

Since strain energy is a scalar quantity, we may add the strain energy of portion AD to that of DB to obtain the total strain energy of the beam.

$$U = U_{AD} + U_{DB}$$

$$= \int_0^a \frac{M_1^2}{2EI} dx + \int_0^b \frac{M_2^2}{2EI} dv$$

$$= \frac{1}{2EI} \int_0^a \left(\frac{P_b}{L} x \right)^2 dx + \frac{1}{2EI} \int_0^b \left(\frac{P_a}{L} v \right)^2 dx$$

$$= \frac{1}{2EI} \frac{P^2}{L^2} \left(\frac{b^2 a^3}{3} + \frac{a^2 b^3}{3} \right)$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL^2} (a + b)}$$

$$\text{Since} (a + b) = L$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL}}$$

b. Substituting the values of P, a, b, E, I, and L in the expression above.

$$U = \frac{(200 \times 10^3)^2 \times (900)^2 \times (2700)^2}{6 (200 \times 10^3) \times (104 \times 10^6) \times (3600)} = 5.27 \times 10^7 \text{ KN.m}$$