UNIT 5 PRINCIPAL STRESSES AND STRAINS

Structure

5.1 INTRODUCTION

In Unit 4, you have already been introduced to the simple states of stress. Stress Analysis is an essential requirement in the evaluation of strength, stiffness, deformations and safety of solids so that one may produce functionally efficient and economic designs. There is a large number of ways in which stresses are induced in solids (a few sample ones you have already learnt), which will engage your attention in the subsequent units. In this unit we shall be concerned with the analysis of a given state of stress (expressed in terms of stress components on selected planes) which will have a bearing on the analysis of strength and safety of solid components.

Objectives

After studying this unit, you should be able to

- define six stress components on mutually perpendicular planes at the requisite location,
- describe the principal plane and principal stress,
- identify the plane of maximum shear stress,
- analyse the state of stress in combined bending & shear and combined bending & torsion, and
- describe various theories of failure.

5.2 STATE OF STRESS

From the point of functional utilization of a solid component we may determine the possible loads (forces) to which it may be subjected to, so that its equilibrium, compatibility and stability are satisfied on the whole. But a more critical analysis will imply the satisfaction of equilibrium at each and every point of the solid. The distribution of stresses over the volume of the solid is analysed taking into these requirements. Once such a distribution has been arrived at it will give the state of stress at each and every point in the solid in terms of the stress components. Often one is not interested in the state of stress at each and every point in the solid, but is satisfied with the analysis of the state of stress at the critical locations of the solid. Description of the general state of stress involves the definition of six stress components namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx} on the three mutually perpendicular planes of a small element at the requisite location. However, in the initial stages of the course, it is sufficient to master the concepts with reference to the state of stress in two dimensions. The general state of stress at any point in a two-dimensional element is given by the stress components σ_x, σ_y and τ_{xy} as shown in Figure 4.28. Ofcourse, any element could only be three-dimensional, but the state of stress is two-dimensional due to the absence of any stress components in the pair of z planes. Hence, in considering equilibrium of forces, the dimension of the element in z direction is taken as unity, in whatever units the other two dimensions are expressed.

5.3 NORMAL AND SHEAR STRESSES

You have been already introduced to the concept, definition and description of normal stress and shear stress. In expressing shear stress components we use two subscripts, such as τ_{xy}, τ_{yx}, τ_{yz} etc. Here, the first subscript denotes the direction of normal to the plane and the second subscript denotes the direction in which the stress (its resultant force) is acting. Thus, τ_{xy} is the shear stress in y direction on x plane, i.e. plane normal to x direction. Logically, all the stress components should have double subscripts. However, as direction of the stress and direction of the normal to the plane are identically same in the case of normal stress component, only a single subscript is used, i.e. σ_x really represents σ_{xx} and so on. In the case of a shear stress component, two subscripts are necessary to define it correctly. The second subscript also indicates the plane on which its complementary component is acting.

We have already stated that among normal stresses, tension is considered positive while compression is considered negative. In the case of shear stresses, one of the components tends to rotate the element in the positive, i.e. anticlockwise direction and is considered positive, while its complementary component which tends to rotate the element in the clockwise direction is considered negative. Accordingly, in the state of stress described in Figure 4.28, τ_{xy} is positive, while τ_{yx} is negative. This definition helps us to determine the sign of the shear stress on inclined planes also.

5.4 STRESS COMPONENTS ON AN ARBITRARY PLANE

You have already studied in Section 4.7 as how to determine the normal and shear stress components on any arbitrary plane whose inclination is defined by the aspect angle θ and Eqs. (4.31) and (4.32) furnish expressions for these components. Let us have an example of practical application.

Example 5.1

Figure 5.1 shows the projection of a rectangular prism *ABCD*, formed by adhesive bonding of two triangular prisms *ABC* and *ACD*. The state of stress in the prism is given by the components $\sigma_x = 40$ N/mm^2, $\sigma_y = 0$ and $\tau = 0$.

If the tensile and shear strengths of the adhesive are 10 N/mm^2 and 12 N/mm^2, verify the safety of the joint and find out the value of σ_x at which the joint will fail.

Solution

The aspect angle θ of the bonding plane *AC*

$$= 90^\circ + \tan^{-1}\left(\frac{50}{75}\right) = 90^\circ + 33.69^\circ = 123.69^\circ$$

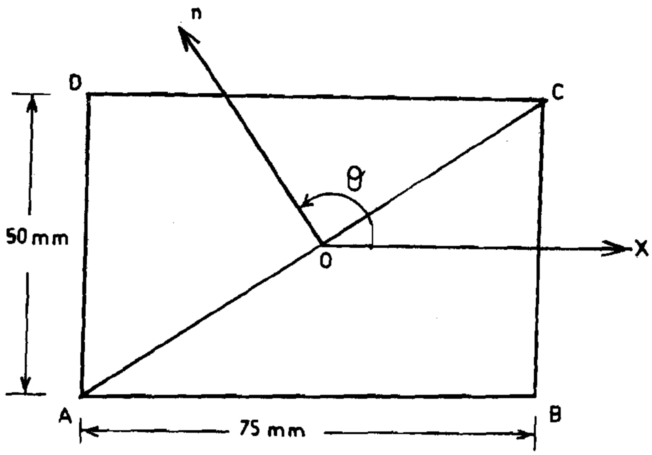

Figure 5.1

Known stress components are as follows :

$\sigma_x = 40$ N/mm^2, $\sigma_y = 0$ and $\tau_{xy} = 0$

Stress components on plane *AC*,

$$\text{Normal stress, } \sigma_n = \sigma_x \cos^2 \theta$$

$$= 40 \cos^2 123.69^\circ$$

$$= 6.14765 \text{ N/mm}^2 < 10 \text{ N/mm}^2$$

$$\text{Shear Stress } \tau_{nt} = -\frac{\sigma_x}{2} \sin 2\theta$$

$$= -\frac{40}{2} \sin (2 \times 123.69^\circ)$$

$$= -14.428 \text{ N/mm}^2 > 12 \text{ N/mm}^2$$

The tensile stress on plane *AC* is well within the tensile strength of the bond. But the shear stress on the plane exceeds the shear strength of the bond and hence, the bond will fail in shear.

Let us find the normal stress σ_x that may be safely applied.

Shear strength of the bond = 12 N/mm^2

$$\text{Shear stress on bonding plane} = \frac{-\sigma_x}{2} \sin 2\theta$$

$$\therefore \quad -12 = \frac{-\sigma_x}{2} \sin 247.38^\circ$$

$$\sigma_x = \frac{12 \times 2}{\sin 247.38^\circ} = 33.27 \text{ N/mm}^2$$

So the maximum stress we may apply on the plane *CB* is 33.27 N/mm^2. Here, you may note that in strength analysis the sign of the shear stress has no significance, while the sign of the normal stress is important, since the tensile and compressive strengths may differ considerably.

SAQ 1

If the prism shown in Figure 5.1 is bonded along the diagonal *DB*, instead of *AC*, verify the safety of the joint and calculate the magnitude of σ_x at which the joint will fail.

5.5 PRINCIPAL STRESSES AND PRINCIPAL PLANES

5.5.1 Definition

In Section 5.4, we have seen that for a given state of stress at a point, the magnitude of normal stress and shear stress may vary with respect to the inclination of planes. If we are concerned with the safety of solids under stress, we are required to find on which planes extreme values of normal and shear stress components are present. Hence, it is essential to know :

(1) Maximum tensile stress,

(2) Maximum compressive stress, and

(3) Maximum shear stress.

In addition, we may also require to know the planes on which these values occur.

The extreme values of normal stresses are called the **Principal Stresses** and the planes on which the principal stresses act are called the principal planes. In two-dimensional problems, there are two principal stresses, namely the **major principal stress** and the **minor principal stress** which are defined as the maximum and minimum values of the normal stresses respectively. Here, the maximum or minimum is to be considered algebraically. For example, if the principal stresses happen to be 20 N/mm^2 tensile and 75 N/mm^2 compressive, the tensile stress of 20 N/mm^2 is to be taken as the major principal stress denoted by the symbol σ_1 and the compressive stress of 75 N/mm^2 is to be taken as the minor principal stress (algebraically –75 N/mm^2) and denoted by the symbol σ_2 . The corresponding planes are defined as major and minor principal planes.

5.5.2 Expressions for Principal Planes and Principal Stresses

In calculus, you have learnt that when a function reaches maximum or minimum its derivative with respect to the independent variable becomes zero. Since the normal stress on an arbitrary plane is a function of the aspect angle θ as given by the expression,

$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$, the maxima and minima of σ_n will occur on the planes for which $\frac{d\sigma_n}{d\theta}$ becomes zero, (similarly, τ_{nt} will be maximum on planes where $\frac{d\tau_{nt}}{d\theta} = 0$).

Let us now derive the expression,

$$\frac{d\sigma_n}{d\theta} = \frac{\sigma_x - \sigma_y}{2}(-2 \sin 2\theta) + \tau_{xy}\, 2 \cos 2\theta$$

$$= 2\left(\tau_{xy} \cos 2\theta + \frac{\sigma_y - \sigma_x}{2} \sin 2\theta\right)$$

$$= 2\,\tau_{nt}$$

i.e.,
$$\frac{d\sigma_n}{d\theta} = 2\,\tau_{nt} \qquad (5.1)$$

Eq. (5.1) gives an important characteristic of the principal plane, namely, the absence of shear stress components on the plane. We can, therefore, alternatively define a principal plane as a plane on which only a normal stress component is acting. When dealing with a three-dimensional state of stress you will find that the third principal plane is neither maximum nor minimum. Hence, we will define principal planes as planes on which shear stresses are zero.

Equating $\frac{d\sigma_n}{d\theta}$ to zero, we get

$$\tau_{xy} \cos 2\theta + \frac{\sigma_y - \sigma_x}{2} \sin 2\theta = 0$$

or

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{-2\,\tau_{xy}}{\sigma_y - \sigma_x} = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}$$

Denoting the specific angles defining principal planes by ϕ_1 and ϕ_2,

or $$\tan 2\phi = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} \tag{5.2}$$

Eq. (5.2) gives a condition for the determination of principal planes. Eq. (5.2) will have two solutions within the range $-\pi/2 < \phi < \pi/2$ and they will give the orientation of principal planes.

Further, the second derivative, $\frac{d^2\sigma_n}{d\theta^2}$ will be negative for the solution ϕ_1 (aspect angle of the major principal plane) and positive for the solution ϕ_2 (aspect angle of the minor principal plane). Let us obtain these expressions too.

$$\frac{d^2\sigma_n}{d\theta^2} = \frac{d}{d\theta}\left(\frac{d\sigma_n}{d\theta}\right) = -2\left(\frac{\sigma_y - \sigma_x}{2}\,2\cos 2\theta - 2\,\tau_{xy}\sin 2\theta\right).$$

$$\frac{d^2\sigma_n}{d\theta^2} = -4\left(\frac{\sigma_y - \sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta\right) \tag{5.3}$$

After obtaining the solutions ϕ_1 and ϕ_2 of Eq. (5.2), their values may be substituted in the expression for $\frac{d^2\sigma_n}{d\theta^2}$ given in Eq. (5.3) and the major and minor principal planes may be identified. But in practical solutions this step is rarely required.

Instead, substitute the two solutions ϕ_1 and ϕ_2 in the expression for normal stress and obtain the values of principal stresses σ_1 and σ_2 and corresponding principal planes may be identified.

Now, let us derive the general expressions for the principal stresses. Since, we know that $\tan 2\phi = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}$ on a principal plane, we may write as,

$$\sin 2\phi = \frac{\tau_{xy}}{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}} \tag{5.4}$$

and $$\cos 2\phi = \frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}} \tag{5.5}$$

Substituting Eqs. (5.4) and (5.5) in the expression for σ_n, we obtain

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}\right) + \frac{\tau_{xy}\,\tau_{xy}}{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2}{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}} + \frac{\tau_{xy}^2}{\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Since $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ will have two roots namely $\pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$, we may write the final expression for major and minor principal stresses as follows :

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (5.6)$$

Eqs. (5.2) and (5.6) may be used to readily determine the principal planes and principal stresses.

SAQ 2

Derive an expression for the maximum shear stress in a general two dimensional state of stress and also an expression for the aspect angle of the corresponding plane.

Let us now have an example for determination of principal stresses and principal planes, given the state of stress.

Example 5.2

Evaluate the principal stresses and principal planes for the state of stress shown in Figure 5.2.

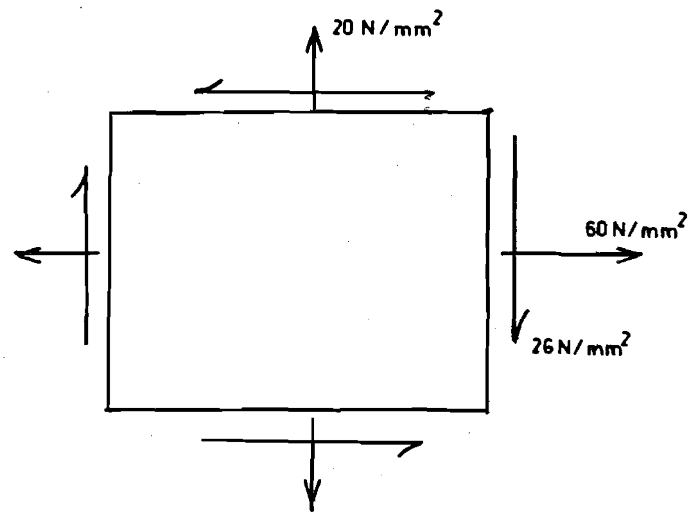

Figure 5.2

Solution

Given $\sigma_x = 60$ N/mm^2

$\sigma_y = 20$ N/mm^2

$\tau_{xy} = -26$ N/mm^2

On substituting in Eq. (5.6), we get

$$\sigma_{1,2} = \frac{60 + 20}{2} \pm \sqrt{\left(\frac{60 - 20}{2}\right)^2 + (-26)^2}$$

$$= 40 \pm 32.8$$

$\therefore \quad \sigma_1 = 72.8$ N/mm^2 and $\sigma_2 = 7.2$ N/mm^2

Again substituting the values σ_x, σ_y and τ_{xy} in Eq. (5.2).

$$\tan 2\phi = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times (-26)}{60 - 20} = -1.3$$

Since θ is general angle, the specific angles representing the principal planes are designated as ϕ_1 and ϕ_2.

$$\therefore\ 2\phi = -52.43^\circ, 127.57^\circ$$

using $2\phi = -52.43^\circ$

$$\sigma_n = \frac{60+20}{2} + \frac{60-20}{2} \cos(-52.43^\circ) - 26 \sin(-52.43^\circ).$$

$$= 72.8 \text{ N/mm}^2$$

Hence, we recognise that $\phi_1 = \frac{-52.43^\circ}{2}$ defines the major principal plane and

therefore, $\phi_2 = \frac{127.57^\circ}{2}$ should define the minor principal plane.

SAQ 3

(a) Evaluate the principal stresses and principal planes for the state of stress shown in Figure 5.3.

(b) Also find the normal and shear stress components on the planes whose aspect angles are given as 30°, 45° and 75°.

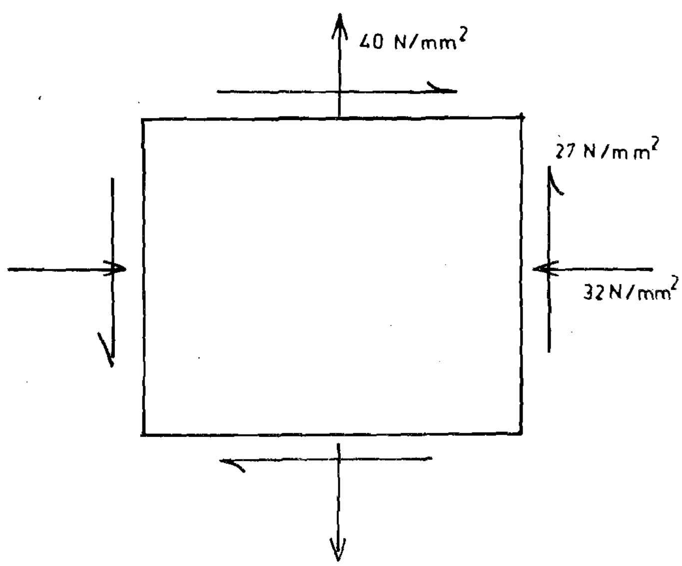

Figure 5.3

5.5.3 Maximum Shear Stress

We have the general expression for shear stress as,

$$\tau_{nt} = \tau_{xy} \cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta$$

Differentiating w.r.t. θ, and equating the derivative to zero,

$$\frac{d\tau_{nt}}{d\theta} = -2\,\tau_{xy} \sin 2\theta - (\sigma_x - \sigma_y) \cos 2\theta = 0$$

$$\therefore\ \tan 2\theta = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}. \qquad (5.7)$$

Since the planes on which maximum shear stresses occur are specific set of planes we may denote them distinctly by ψ (instead of general aspect angle θ).

Comparing Eqs. (5.2) and (5.7), we conclude that $2\psi = 2\phi \pm 90^\circ$ as $\tan 2\phi \cdot \tan 2\psi = -1$.

$$\therefore \quad \psi = \phi \pm 45^\circ \tag{5.8}$$

Eq. (5.8) indicates that the planes of maximum shear stress bisect the right angles between the major and minor principal planes.

The normals to the major and minor principal planes may now be defined as the major and minor principal axes. Once the principal stresses and principal planes are known, further analysis may be simplified by expressing the state of stress w.r.t. a new coordinate system with major and minor principal axes as coordinate axes themselves. These axes are usually called axes 1 and 2 respectively.

The general expressions for stress components on arbitrary planes whose aspect angle $\bar{\theta}$ may now be measured with axis-1 as reference axis.

Hence,
$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\bar{\theta} \tag{5.9}$$

$$\tau_{nt} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\bar{\theta} \tag{5.10}$$

Eq. (5.8) already defines that $\bar{\theta}$ should be $\pm 45^\circ$ for τ_{nt} to be maximum.

Thus,
$$\tau_{\max,\min} = \frac{\sigma_1 - \sigma_2}{2} \sin (\pm 90^\circ)$$

$$\therefore \quad \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} \tag{5.11}$$

and
$$\boxed{\tau_{\min} = -\frac{\sigma_1 - \sigma_2}{2}.}$$

Since the sign of maximum shear stress is not significant, expression for $\tau_{\min}$ is not generally used. Let us have a few examples.

Example 5.3

The state of stress at a critical point of a strained solid is given by $\sigma_x = 70$ kN/mm^2, $\sigma_y = -50$ N/mm^2 and $\tau_{xy} = 45$ N/mm^2. If the strength of the solid in tension, compression, and shear are given as 120 N/mm^2, 90 N/mm^2 and 75 N/mm^2 respectively, verify the safety of the component.

Solution

$$\text{Given } \sigma_x = 70 \text{ N/mm}^2$$

$$\sigma_y = -50 \text{ N/mm}^2$$

$$\tau_{xy} = 45 \text{ N/mm}^2$$

$$\therefore \quad \sigma_{1,2} = \frac{70 + (-50)}{2} \pm \sqrt{\left(\frac{70 - (-50)}{2}\right)^2 + 45^2}$$

$$= 85, -65 \text{ N/mm}^2$$

$$\text{Maximum shear stress, } \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{85 - (-65)}{2} = 75 \text{ N/mm}^2$$

$$= 75 \text{ N/mm}^2$$

All the stresses are within the strength limits of the solid and hence, the solid is safe.

$$\text{Factor of safety in tension} = \frac{120}{85} = 1.412$$

$$\text{Factor of safety in compression} = \frac{90}{65} = 1.3846$$

$$\text{Factor of safety in shear} = \frac{75}{75} = 1$$

Here, maximum tensile and compressive stresses are well within strength limits, maximum shear stress has reached the strength limit and therefore if the state of stress is proportionally raised the solid will fail in shear.

Example 5.4

A machine component is made of a material whose ultimate strength in tension, compression and shear are 40 N/mm^2, 110 N/mm^2 and 55 N/mm^2 respectively. At the critical point in the component the state of stress is represented by

$$\sigma_x = 25 \text{ N/mm}^2 \text{ and } \sigma_y = -75 \text{ N/mm}^2.$$

Find the maximum value of the shear stress τ_{xy} which will cause failure of the component and also specify the mode of failure.

Solution

Given state of stress : $\sigma_x = 25$ N/mm^2

$$\sigma_y = -75 \text{ N/mm}^2$$

We have to find what τ_{xy} is safe, if $\sigma_1 \leq 40$ N/mm^2, $\sigma_2 \geq -110$ N/mm^2 and $\tau_{max} \not> 55$ N/mm^2.

The above three conditions are to be independently satisfied.

Now,
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \leq 40$$

In the limiting case

$$40 = \frac{25 + (-75)}{2} + \sqrt{\left(\frac{25 - (-75)}{2}\right)^2 + \tau_{xy}^2}$$

$$40 = -25 + \sqrt{50^2 + \tau_{xy}^2}$$

or
$$\tau_{xy}^2 = \left[40 - (-25)\right]^2 - 50^2 = 1725$$

$$\therefore \quad \tau_{xy} = \pm 41.533 \text{ N/mm}^2$$

Note that the limiting case of $\sigma_1 = 40$ N/mm^2 will occur for both the τ_{xy} values of 41.533 N/mm^2 and $-$41.533 N/mm^2. But the planes of failure will be different.

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \geq -110$$

In the limiting case,

$$-110 = \frac{25 + (-75)}{2} - \sqrt{\left(\frac{25 - (-75)}{2}\right) + \tau_{xy}^2}$$

$$-110 = -25 - \sqrt{50^2 + \tau_{xy}^2}$$

$$-85 = -\sqrt{50^2 + \tau_{xy}^2}$$

$$(-85)^2 = 50^2 + \tau_{xy}^2$$

$$\tau_{xy} = \pm\sqrt{85^2 - 50^2} = \pm\ 68.7386 \text{ N/mm}^2$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \leq 55$$

i.e.
$$50^2 + \tau_{xy}^2 = (55)^2$$

$$\tau_{xy} = \pm\ \sqrt{55^2 - 50^2} = \pm\ 22.91 \text{ N/mm}^2$$

The permissible value of τ_{xy} is different for different limiting criteria, namely

$$|\tau_{xy}| \leq 41.53 \text{ if } \sigma_1 \leq 40$$
$$\leq 68.74 \text{ if } \sigma_2 \geq -110$$
$$\leq 22.91 \text{ if } \tau_{max} \not> 55$$

Hence, we find that the maximum safe value of τ_{xy} is only 22.91 N/mm^2 and the material will fail in shearing mode.

Example 5.5

The state of stress at a point in a loaded solid is prescribed on two faces of an element whose shape is a triangular prism as shown in Figure 5.4. Evaluate the principal stresses and principal planes.

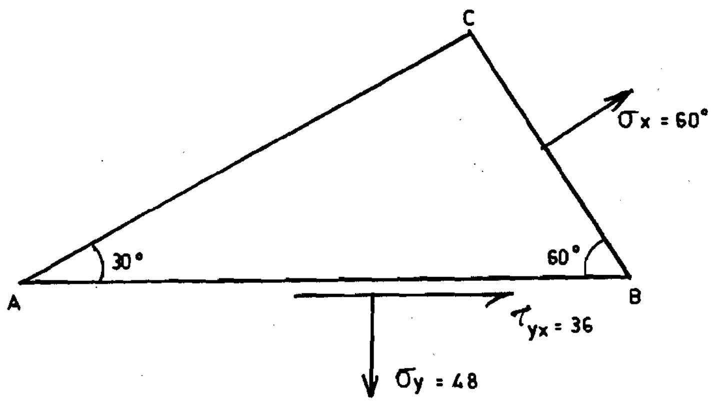

Figure 5.4

Solution

Here, we have to first obtain the value of σ_x and subsequent calculations will be a standard set.

Given $\sigma_y = 48$ MPa

$\tau_{yx} = 36$ MPa $\therefore$ $\tau_{xy} = -36$ MPa

$\sigma_{30^\circ} = 60$ MPa (aspect angle of plane BC is $+ 30^\circ$)

$$\sigma_{30^\circ} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2 \times 30^\circ) + \tau_{xy} \sin(2 \times 30^\circ)$$

$$60 = \frac{\sigma_x + 48}{2} + \frac{\sigma_x - 48}{2} \cos 60^\circ + (-36) \sin 60^\circ$$

$$60 = \frac{\sigma_x}{2} + 24 + \frac{\sigma_x}{2} \cos 60^\circ - 24 \cos 60^\circ - 36 \sin 60^\circ$$

i.e. $$60 = \frac{\sigma_x}{2} (1 + \cos 60^\circ) + 24 - 24 \cos 60^\circ - 36 \sin 60^\circ$$

$$\therefore \sigma_x = \frac{2}{1 + 0.5} [\, 60 - 24 - (-24 \times 0.5) - (-36) \times 0.866 \,]$$

$$= \frac{2}{1.5} [\, 79.177 \,] = 105.57 \text{ MPa}$$

$$\sigma_{1,2} = \frac{105.57 + 48}{2} \pm \sqrt{\left(\frac{105.57 - 48}{2}\right)^2 + (-36)^2} = 76.835 \pm 46.093$$

$$\therefore \sigma_1 = 76.835 + 46.093 = 122.928 \text{ MPa}$$

$$\sigma_2 = 76.835 - 46.093 = 30.742 \text{ MPa}$$

Let the aspect angles of the principal planes be ϕ

$$\tan 2\phi = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times (-36)}{105.57 - 48} = -51.35^\circ \text{ or } 128.65^\circ$$

$$\therefore \phi = -25.675^\circ \text{ or } 64.325^\circ$$

Substituting $\phi = -25.675°$ (or $2\phi = -51.35°$) in expression for σ_n

$$\sigma_n = \frac{105.57 + 48}{2} + \frac{105.57 - 48}{2} \cos(-51.35°) - 36 \sin(-51.35°)$$

$$= 122.92 \text{ MPa}$$

$$\therefore \ \phi_1 = -25.675° \text{ and } \phi_2 = 64.325°$$

SAQ 4

(a) In the element ABC shown in Figure 5.4, find the normal and shear stress components on the plane AC.

(b) If the state of stress at a point is defined by the stress component $\sigma_x = 9$ MPa, $\sigma_y = -7$ MPa and $\tau_{xy} = 5$ MPa, find the principal stresses and principal planes. Also find the plane on which normal and shear stress components are equal in magnitude.

5.6 CIRCULAR REPRESENTATION OF STATE OF STRESS

From a state of stress defined by the components σ_x, σ_y and τ_{xy}, we express the stress components on an arbitrary plane as

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad (4.31)$$

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta \qquad (4.32)$$

which we may rewrite in general as

$$\sigma = a + b \cos 2\theta + c \sin 2\theta \qquad \text{(i)}$$

$$\tau = -b \sin 2\theta + c \cos 2\theta \qquad \text{(ii)}$$

Let us now try to establish direct relationship between σ and τ by eliminating θ between Eqs. (i) and (ii).

To simplify the effort, let us take the origin of coordinates at $(a, 0)$, so that the new variable $\bar{\sigma} = (\sigma - a)$ is considered for developing the relationship.

$$\bar{\sigma} = b \cos 2\theta + c \sin 2\theta \qquad \text{(iii)}$$

$$\tau = -b \sin 2\theta + c \cos 2\theta \qquad \text{(iv)}$$

Squaring and adding, we get

$$\bar{\sigma}^2 + \tau^2 = b^2 \cos^2 2\theta + c^2 \sin^2 2\theta + 2bc \cos 2\theta \,.\, \sin 2\theta$$
$$+ b^2 \sin^2 2\theta + c^2 \cos^2 2\theta - 2bc \cos 2\theta \sin 2\theta$$
$$= b^2 (\cos^2 2\theta + \sin^2 2\theta) + c^2 (\sin^2 2\theta + \cos^2 2\theta)$$

i.e. $\bar{\sigma}^2 + \tau^2 = b^2 + c^2.$

Since b and c are constants let $b^2 + c^2 = r^2$.

$$\therefore \ \bar{\sigma}^2 + \tau^2 = r^2 \qquad (5.12)$$

Eq. (5.12) shows that if, σ and τ, the normal and shear stress components on any arbitrary plane are plotted as coordinates, the locus of the point will be a circle whose centre will be $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and radius will be $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ $\left(\text{i.e. } \sqrt{b^2 + c^2}\right)$

In other words, the state of stress at any point may be represented by a circle and a point on the circle represents the normal and shear stress components, on some plane as horizontal and vertical coordinates.

Let us now consider the circular representation more closely so as to obtain clear interpretation of the state of stress.

In Figure 5.5 a coordinate system (σ, τ) has been formed with σ on horizontal axis and τ on the vertical downward axis. On this coordinate plane or $\sigma - \tau$ plane a point X is choosen whose coordinates are σ_x and τ_{xy} respectively. Also another point Y is choosen with coordinates σ_y and τ_{yx} (or σ_y and $-\tau_{xy}$).

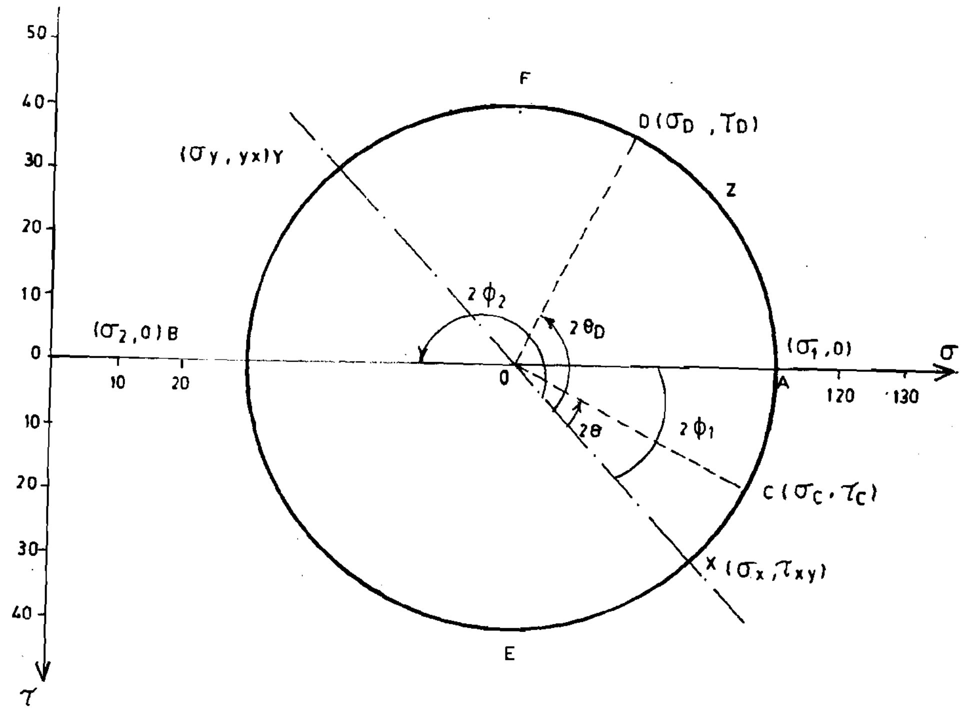

Figure 5.5 : Circular Representation of State of Stress

These two points X and Y represent the stress components on two planes and hence, according to Eq. (5.12), should be points on a circle. Point X represents x plane and point Y represents the y plane. When we join the points X and Y by a straight line, we find that the line passes through the point $\bar{O}$ on the σ axis which should be the origin of the circle defined by Eq. (5.12). As $\bar{O}$ is the origin of circle and X and Y are points on the periphery of the circle the line $X\bar{O}Y$ (or simply XY) should be the diameter of the circle which we are trying to establish. Hence, draw a circle with XY as diameter.

Now any point on this circle should represent the state of stress on an oblique plane. Let us consider a few specific points. Points A and B lie on the horizontal (σ) axis, i.e. $\tau = 0$ and hence they represent the two principal planes. Since σ is maximum at A, point A represents the major principal plane (and hence, σ coordinate of A is σ_1) and point B represents the minor principal plane. $\angle X\bar{O}A$ and $\angle X\bar{O}B$ are equal to $2\phi_1$ and $2\phi_2$ respectively where ϕ_1 and ϕ_2 are the inclinations of the major and minor principal planes with the x plane. Consider any arbitrary point say C such that the angle $\angle X\bar{O}C$ is equal to $2\theta_c$. Then, the coordinates of the point C give the normal and shear stress components on a plane inclined at θ_c to the x plane. Similarly, each point, Z on this circle may be interpreted to give the normal and shear stress components on a plane whose inclination with x axis is $\frac{1}{2} \angle X\bar{O}Z$.

5.7 MOHR'S CIRCLE FOR THE ANALYSIS OF STATE OF STRESS

The circle in Figure 5.5 is called the Mohr's Circle of stress. Mohr's circle is very useful in graphical analysis of state of stress at a point.

Given the state of stress (defined by σ_x, σ_y and τ_{xy}), the procedure for construction of the Mohr's circle was discussed in Section 5.6. The determination of principal stresses and principal planes from the Mohr's circle was also indicated. In this section let us learn a few applications of stress analysis with the help of the Mohr's circle.

Suppose we need to find the normal and shear stress components on a plane whose inclination to x plane is θ_D. We need only to draw a radial line OD making an angle $2\theta_D$ with the radial line OX. The coordinates (σ_D, τ_D) of the point D will give the normal and shear stress components on the plane. Thus, once the Mohr's circle is constructed, the stress components on any plane may be readily obtained.

Example 5.6

The state of stress at a point is given by the stress components $\sigma_x = 70$ MPa, $\sigma_y = 10$ MPa and $\tau_{xy} = -40$ MPa. Using Mohr's circle find (i) Principal stresses, and (ii) Principal planes. Also, determine the normal and shear stress components on planes making 25°, 40° and 60° respectively with the x plane.

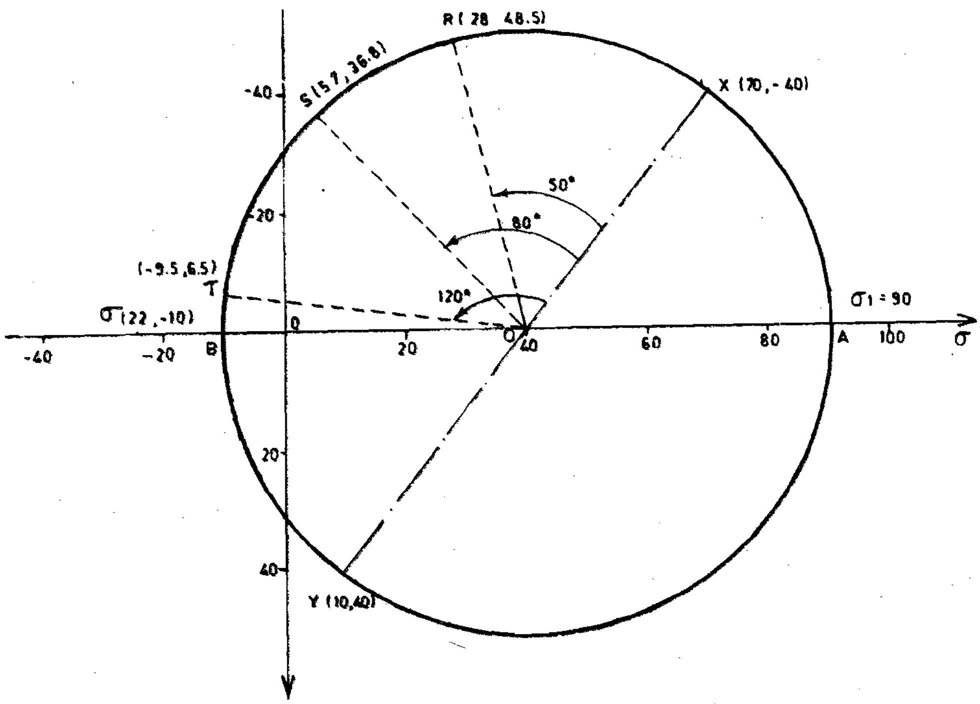

Figure 5.6

Solution

(1) Choose (σ, $-\tau$) coordinate system to a suitable scale.

(2) Mark the points X (70, − 40) and Y (10, + 40).

(3) Draw a circle with XY as diameter. This circle cuts σ axis at A and B.

(4) Measure the coordinates of A and B to obtain principal stresses

σ_1 = 90 MPa, σ_2 = −10 MPa

(The radius of the Mohr's circle gives τ_{max} = 50 MPa)

(5) Measure $\angle XOA$; Here, $\angle XOA$ = −52.8°.

∴ Aspect angle ϕ_1 of major principal plane is −26.4°.

(6) Measure $\angle XOB$; Here, $\angle XOB$ = + 127.2°.

∴ Aspect angle ϕ_2 of minor principal plane is + 63.6°.

(7) Draw radial lines OR, OS and OT making angles 50°, 80° and 120° with OX.

(8) Coordinates of R give the normal and shear stress components on the plane which makes 25° with the x plane. Here,

σ_n = 28 MPa; τ_{nt} = − 48.5 MPa.

(9) Coordinates of S give the normal and shear stress components on the plane with aspect angle 40°. Here,

$$\sigma_n = 5.7 \text{ MPa}; \ \tau_{nt} = -36.8 \text{ MPa}$$

(10) Coordinates of T give the normal and shear stress components on the plane with aspect angle 60°. Here,

$$\sigma_n = -9.5 \text{ MPa}; \ \tau_{nt} = -6.5 \text{ MPa}.$$

SAQ 5

Analyse the states of stress shown in Figure 5.7 and find the principal stresses and principal planes.

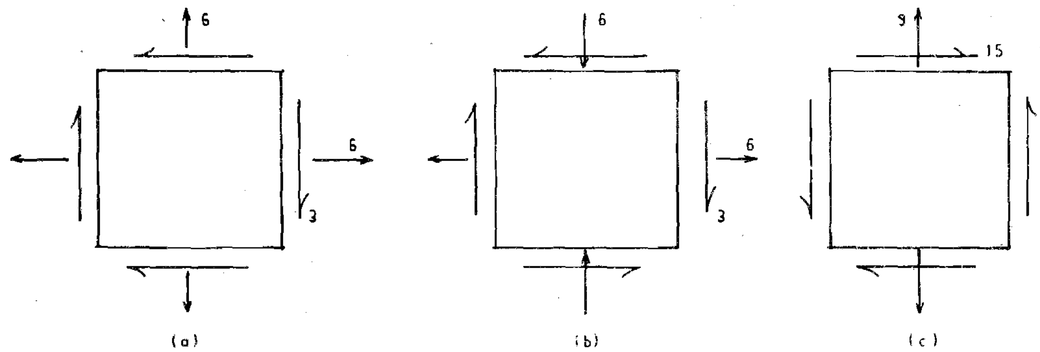

Figure 5.7

(a) Verify the theoretical results by drawing Mohr's circles for each case.

(b) Verify the results of Example 5.6 analytically.

(c) Solve the Examples 5.2, 5.3 and 5.4 using Mohr's circle and verify the results.

(d) Draw a Mohr's circle to represent the state of stress $\sigma_x = \sigma_y = 0$ and $\tau_{xy} = 10$ and then, find σ_1 and σ_2.

(e) Draw a Mohr's circle to represent the state of stress $\sigma_x = \sigma_y$ and $\tau_{xy} = 0$.

5.8 STATE OF STRESS IN COMBINED BENDING AND SHEAR

So far we have been analysing the state of stress defined by the stress components σ_x, σ_y and τ_{xy}. But we have not directed our attention as to how different stress components are induced and how they vary within the solid body and also how to identify the critical locations. In this and the next section, we shall consider two specific types of loading and identify the pattern of stresses induced in these cases.

You would have seen a large number of girders supporting bridges and still a large number of beams supporting roof or floor slabs in buildings. You will learn, in later Units 6, 7 and 8, how these members are loaded and supported and how the stresses induced in them are calculated. Here we shall consider a few simple cases.

Consider a beam of rectangular cross section as shown in Figure 5.8. On any section of the beam there may be acting two kinds of reactive resistance, namely, a Bending Moment M and a Shearing Force Q. In terms of the cross sectional dimensions (breadth 'b' and height 'h') the stress components on a layer located at a height 'y' from the neutral axis (centroidal axis here) are given as follows :

$$\sigma_x = -\frac{12\,My}{bd^3} \qquad \text{(i)}$$

$$\sigma_y = 0 \quad \text{(ii)}$$

$$\tau_{xy} = \frac{1.5Q}{bh}\left(1 - \frac{4y^2}{h^2}\right) \quad \text{(iii)}$$

The variations of stresses across the depth as per Eqs. (i) and (iii) are shown in Figures 5.8 (ii) and (iii). (How Eqs. (i) to (iii) are obtained and how to obtain corresponding expressions for beams or girders of other cross sections will be dealt with in detail in Units 7 and 8.)

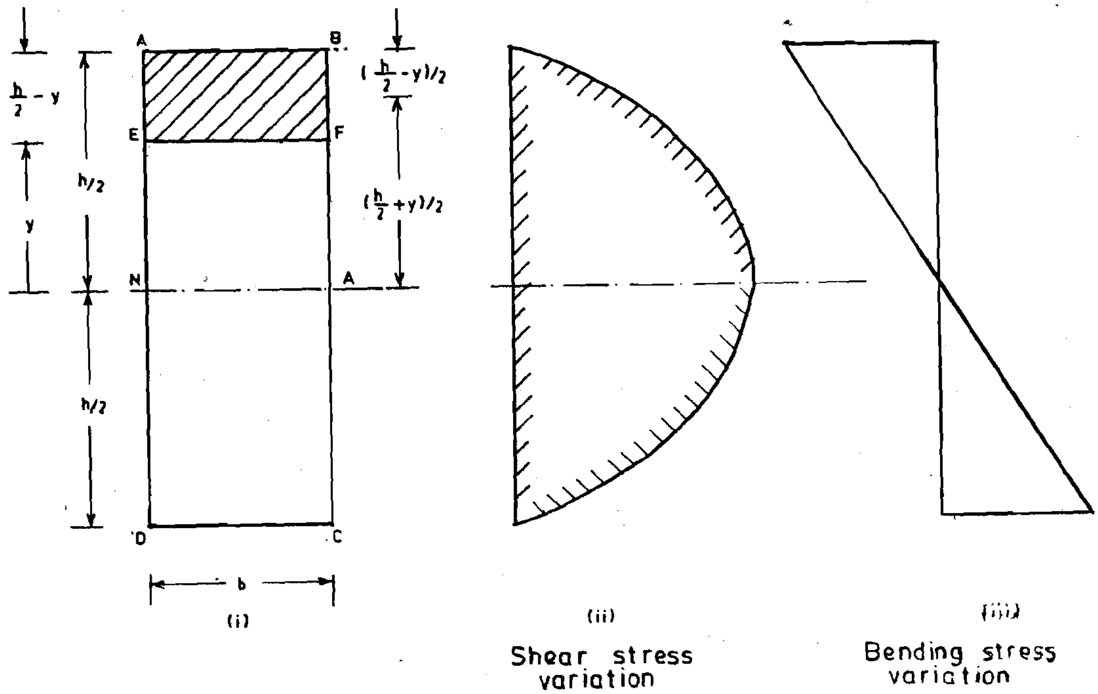

Figure 5.8

Example 5.7

For the cross section shown in Figure 5.8, let us take $b = 100$ mm, $d = 300$ mm and $M = 9 \times 10^7$ N mm and $Q = 18 \times 10^4$ N and analyse the state of stress at a layer 50 mm below the top layer.

Solution

Using Eqs. (i) and (iii), $$\sigma_x = \frac{-9 \times 12 \times 10^7 \times y}{100 \times 300^2}$$

Since extreme layers are at 150 mm from neutral axis,

$$y = 150 - 50 = 100$$

$$\therefore \ \sigma_x = \frac{-9 \times 12 \times 10^7 \times 100}{100 \times 300^2} = -40 \text{ N/mm}^2$$

$$\tau_{xy} = \frac{1.5 \times 18 \times 10^4}{100 \times 300}\left(1 - \frac{4 \times 100^2}{300^2}\right) = 5 \text{ N/mm}^2$$

Hence, $$\sigma_{1,2} = -\frac{40}{2} \pm \sqrt{\left(-\frac{40}{2}\right)^2 + 5^2}$$

i.e., $$\sigma_2 = -40.616 \text{ N/mm}^2$$

$$\sigma_1 = +0.616 \text{ N/mm}^2$$

$$\tau_{max} = 20.616 \text{ N/mm}^2.$$

Example 5.8

A beam of I-section as shown in Figure 5.9 (a) is subjected to a bending moment of 84.928 kN m and a shear force of 106.1 kN. Examine the state of stress at the junction of flange and web.

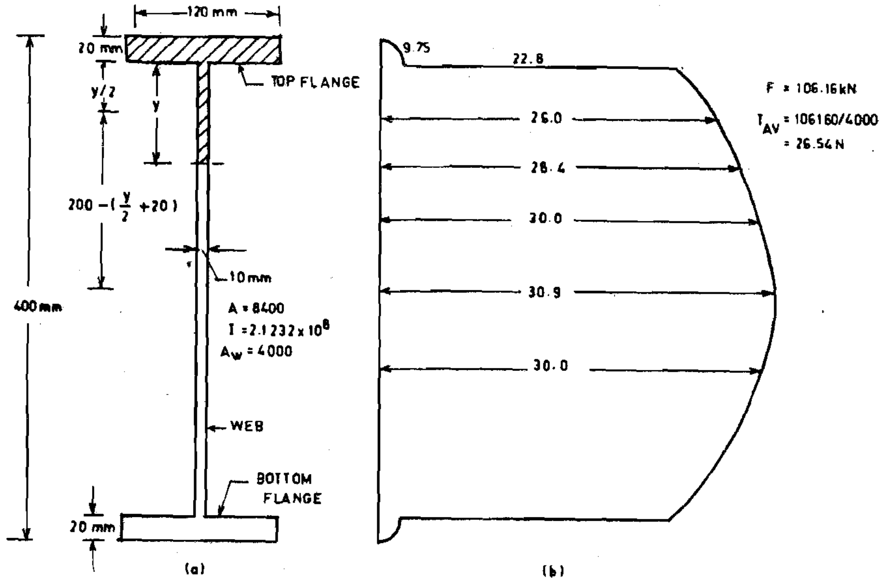

Figure 5.9

Solution

(In this case also the solution will be provided by stating the expressions for stress components, without their derivation which will be dealt with in Units 7 and 8.)

The neutral axis of the section will be the horizontal axis through the centroid of the section.

The bending stress at a layer 'y' above the neutral axis is given as,

$$\sigma_x = \frac{-My}{I} \qquad \text{(iv)}$$

where, I is the moment of inertia of the section and M is the bending moment on the section.

The shear stress at any layer is given as, $\tau_{xy} = \dfrac{Q \,.\, A\bar{y}}{I\,b}$ (v)

where, $A\bar{y}$ is the moment of the area of that portion of the section above or below the layer about the neutral axis, b is the breadth of the layer and Q, the shear force at the section.

For the cross section given, the moment of inertia is given as 2.1232×10^8 mm^4.

Total depth of beam = 400 mm.

$\therefore$ Neutral axis is at a depth of 200 mm from top.

The distance of the layer at junction of web and flange from the neutral axis is 180 mm.

Determination of Bending Stress

$$\sigma_x = \frac{-M\,y}{I} = \frac{-84.928 \times 10^6 \times 180}{2.1232 \times 10^8} = -72 \text{ N/mm}^2$$

Determination of Shear Stress

In this case $A\bar{y}$ will be given by the moment of the flange area about the neutral axis.

$$A\bar{y} = 120 \times 20 \times \left(180 + \frac{20}{2}\right) = 4.56 \times 10^5 \text{ mm}^3$$

$$\therefore\ \tau_{xy} = \frac{106.1 \times 10^3 \times 4.56 \times 10^5}{2.1232 \times 10^8 \times 10} = 22.8 \text{ N/mm}^2$$

$$\sigma_{1,2} = \frac{-72}{2} \pm \sqrt{\left(\frac{-72}{2}\right)^2 + (22.8)^2}$$

Thus, we get, $\sigma_1 = 6.613 \text{ N/mm}^2$

$$\sigma_2 = -78.613 \text{ N/mm}^2$$

$$\tau_{max} = 42.613 \text{ N/mm}^2$$

You may observe here that the junction between flange and web will become the critical zone, when the shear force at the section is considerable. However, experience will tell you, where to look for critical zones in different cases.

5.9 STATE OF STRESS IN COMBINED BENDING AND TORSION

In Section 5.8, we have considered the effect of Bending Moment, a moment which tends to bend the axis of the beam, in terms of stresses produced by it. Another type of moment which tends to rotate the member about its axis is called twisting moment or torque and it produces shear stresses in the member. This phenomenon is called **Torsion.** Study of stresses and strains due to torsion is very difficult except for members with circular (solid or hollow) cross section. If a torque of magnitude T is applied on a circular bar, the shear stress produced at a point located at a radial distance 'r' from the axis of the shaft is given by

$$\tau = \frac{T r}{J} \qquad \text{(vi)}$$

where, J is the polar moment of Inertia of the cross section of the bar.

When a member of circular section is subjected to both bending moment and torque, the maximum stresses due to both load-cases are produced only in extreme fibres and hence, locating the critical section is not a problem. Let us illustrate with a numerical example.

Example 5.9

A prismatic bar of circular section with 80 mm diameter is subjected to a Bending Moment of 5 kN-m and a Torque of 7 kN-m. Analyse the state of stress at the critical section.

Solution

Moment of Inertia, $I = \dfrac{\pi D^4}{64} = \dfrac{\pi \times 80^4}{64} = 2.016 \times 10^6 \text{ mm}^4.$

Polar moment of Inertia, $J = \dfrac{\pi D^4}{32} = \dfrac{\pi}{32} \times 80^4 = 4.032 \times 10^6 \text{ mm}^4.$

Maximum bending stress, $\sigma_x = \dfrac{M y_{max}}{I} = \dfrac{5 \times 10^6 \times 40}{2.016 \times 10^6} = 99.206 \text{ N/mm}^2$

Maximum shear stress, $\tau_{xy} = \dfrac{7 \times 10^6 \times 40}{4.032 \times 10^6} = 69.444 \text{ N/mm}^2.$

Determination of Principal Stress

$$\sigma_{1,2} = \frac{99.206}{2} \pm \sqrt{\left(\frac{99.206}{2}\right)^2 + (69.44)^2}$$

Thus, we get, $\sigma_1 = 134.944 \text{ N/mm}^2$ and $\sigma_2 = -35.737 \text{ N/mm}^2$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 85.34 \text{ N/mm}^2$$

Note

When the cross section of the bar is hollow circular with outer and inner diameters D and d respectively the analysis of stresses is to be carried by the same procedure except for using the expressions,

$$I = \frac{\pi}{64}(D^4 - d^4) \text{ and } J = \frac{\pi}{32}(D^4 - d^4)$$

SAQ 6

A prismatic bar of hollow circular cross-section has the outer and inner diameters as 100 mm and 60 mm respectively. Find the maximum stresses induced due to the action of a bending moment of 5 kN-m along with a torque of 7 kN-m.

Example 5.10

A prismatic bar of hollow circular cross section with outer and inner diameters 100 mm and 80 mm respectively, carries a bending moment of 5 kN-m. If the tensile, compressive and shear strengths of the material are given as 140 N/mm^2, 125 N/mm^2 and 95 N/mm^2 respectively, what is the magnitude of torque that may safely be applied in addition to the bending moment.

Solution

Here, the criteria to be considered are as follows :

$$\sigma_1 \ngtr 140 \text{ N/mm}^2$$

$$\sigma_2 \nless -125 \text{ N/mm}^2$$

$$\tau_{max} \ngtr 95 \text{ N/mm}^2$$

As the magnitudes of σ_1 and σ_2 for such bars will be equal, we need to satisfy only,

$$\sigma_2 \nless -125 \text{ N/mm}^2$$

and
$$\tau_{max} \ngtr 95 \text{ N/mm}^2$$

$$I \text{ for the section} = \frac{\pi}{64}\,(100^4 - 80^4) = 2.898 \times 10^6 \text{ mm}^4$$

$$J \text{ for the section} = \frac{\pi}{32}\,(10^4 - 80^4) = 5.796 \times 10^6 \text{ mm}^4$$

Maximum compression due to bending moment,

$$= -\frac{M}{I}\,y_{max} = \frac{-5 \times 10^6 \times 50}{2.898 \times 10^6} = -86.266 \text{ N/mm}^2$$

If T be the torque applied (in kN-m units), then,

$$\tau_{max} = \frac{T \times 10^6 \times 50}{5.796 \times 10^6} = 8.62664T \text{ N/mm}^2 \qquad \text{(under torsion alone)}$$

$$\sigma_2 = \frac{-86.266}{2} - \sqrt{\left(\frac{-86.266}{2}\right)^2 + (8.62664\,T)^2} = -125$$

$$\therefore \quad -\sqrt{\left(\frac{-86.266}{2}\right)^2 + 8.62664^2 T^2} = -125 + \frac{86.266}{2} = -81.867$$

$$43.133^2 + 8.62664^2 T^2 = (-81.867)^2$$

i.e.
$$T = \left(\frac{81.867^2 - 43.133^2}{8.62664^2}\right)^{1/2} = 8.066 \text{ kN-m.}$$

If the applied torque is within 8.066 kN-m, we are certain that the bar will be safe in compression as well as tension.

We should now analyse what torque will have to be applied if $\tau_{max} \ngtr 95$ N/mm^2.

We know that, $\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ as $\sigma_y = 0$.

$$\tau_{max} = \sqrt{\left(\frac{86.266}{2}\right)^2 + (8.62664T)^2} = 95 \quad \text{(under combined bending \& torsion)}$$

$$T = \left(\frac{95^2 - \left(\frac{86.266}{2}\right)^2}{8.62664^2}\right)^{1/2} = 9.8119 \text{ kN-m}$$

However, we cannot apply this much torque, since it will cause compression failure. Thus, the safe value of additional torque should be restricted to 8.066 kN-m.

SAQ 7

A shaft is to be made of a material with safe strength values in tension, compression and shear of 125 N/mm^2, 105 N/mm^2 and 84 N/mm^2 respectively. What should be the diameter of the shaft to carry a bending moment of 6 kN-m along with a torque of 8.5 kN-m.

5.10 STRAIN ENERGY DUE TO NORMAL STRESS

In Unit 4, you have learnt a few of the characteristics of elastic solids. Here, we shall learn one more important characteristic. Whenever forces are applied on elastic (deformable) solids, the points of application of the forces move due to deformations in the solid, and hence they do work, loosing their potential energy in the process. In elastic solids this energy is fully stored and released when the strains are removed. This stored energy is called **Strain Energy**.

Let us now develop an expression for the strain energy stored in a solid, when it is subjected to a normal stress, say σ_x.

Consider a small element of dimensions *dx*, *dy* and *dz* as shown in Figure 5.10, and subjected to a normal stress σ_x which produces an elongation $d\delta$ in the direction of the stress.

If *E* is the Young's Modulus of the material,

strain produced, $\varepsilon_x = \frac{\sigma_x}{E}$.

$\therefore$ elongation $d\delta = dx \,.\, \varepsilon_x = dx \,.\, \frac{\sigma_x}{E}$

Total force applied on the element *dF*, is given by the stress multiplied by area on which it is applied.

i.e. $dF = \sigma_x \,.\, dy \,.\, dz$

$\therefore$ Work done by dF = Force × Average Displacement

i.e. $dU = dF \,.\, \frac{d\delta}{2}$

$$dU = \frac{1}{2}\sigma_x \,.\, dy \,.\, dz \,.\, \frac{\sigma_x}{E} dx$$

$$dU = \frac{1}{2E} \,.\, \sigma_x^2 \,.\, dx \,.\, dy \,.\, dz$$

Since $dx \,.\, dy \,.\, dz$ represents the volume of the element dv,

$$dU = \frac{\sigma_x^2}{2E} dv \qquad (5.13)$$

(By similar reasoning, due to other normal stress components the energy stored in the element may be shown to be $\frac{\sigma_y^2}{2E} dv$ and $\frac{\sigma_z^2}{2E} dv$.)

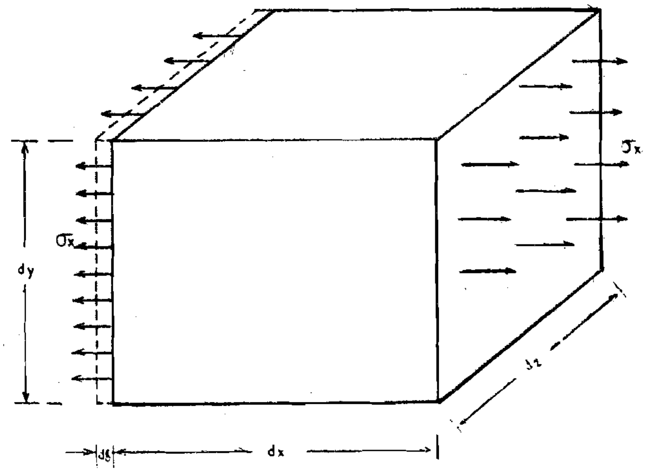

Figure 5.10

Total energy stored in the solid,

$$U = \int_v \frac{\sigma_x^2}{2E}\, dv \tag{5.14}$$

Dividing Eq. (5.13) by dv, we can also obtain an expression for the strain energy density at any point as,

$$u = \frac{\sigma_x^2}{2E} \tag{5.15}$$

(Also the strain energy density due to other stress components may be shown to be $\frac{\sigma_y^2}{2E}$ and $\frac{\sigma_z^2}{2E}$ respectively.)

The application of Eqs. (5.13), (5.14) and (5.15) will be dealt elaborately in a later unit. However, these expressions are presented here, as they are useful in the study of a given state of stress and will be explained in Section 5.14.

5.11 STRAIN ENERGY DUE TO SHEAR STRESS

A small element of dimensions dx, dy and dz is subjected to a shear stress component τ_{xy} and undergoing shear strain γ_{xy} is shown in Figure 5.11.

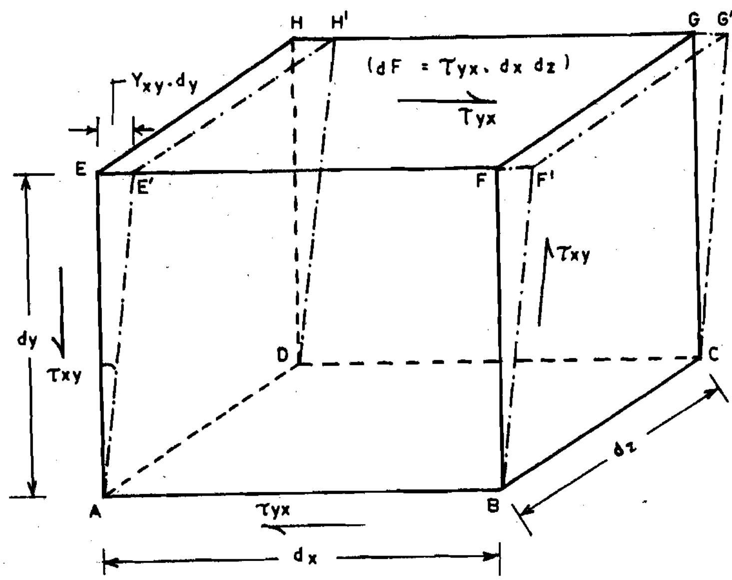

Figure 5.11

As shown in Section 5.10, we may calculate the work done as product of force dF and average displacement $\frac{d\delta}{2}$.

$$\text{Displacement,} \quad d\delta = \gamma_{xy}\, dy$$

$$\text{Force,} \quad dF = \tau_{xy}\, dx\, dy$$

$$\text{Shear strain,} \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\therefore \; d\delta = \frac{\tau_{xy}}{G}\, dy$$

$\therefore$ Work done or energy stored in the element,

$$dU = \frac{1}{2}\, \tau_{xy}\, dx\, dz \,.\, \frac{\tau_{xy}}{G} \,.\, dy$$

$$dU = \frac{1}{2}\, \frac{\tau_{xy}^2}{G} \,.\, dv \qquad (5.16)$$

$\therefore$ Total Strain Energy

$$U = \int_v \frac{\tau_{xy}^2}{2G}\, dv \qquad (5.17)$$

Strain Energy Density at any point,

$$u = \frac{\tau_{xy}^2}{2G} \qquad (5.18)$$

Expressions similar to Eqs. (5.16), (5.17) and (5.18) can be developed for other shear stress components τ_{yz} and τ_{zx} also.

5.12 STRAIN ENERGY IN TERMS OF PRINCIPAL STRESSES

In Sections 5.10 and 5.11, we have derived expressions for strain energy assuming that at a time only one stress component is acting. For example, we have taken ε_x as equal to $\frac{\sigma_x}{E}$ which is true only if σ_x alone is acting. Otherwise, we know that,

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E},$$

and the use of such expressions will result in more complex expressions for strain energy and strain energy density. Hence, in the case of a general stress field, we may be able to get simpler expressions if we reduce the number of terms to be considered. A general stress field with six components of stress namely σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx} may be expressed in terms of equivalent principal stress components σ_1, σ_2 and σ_3. In terms of these three components, let us now derive expressions for total strain energy.

5.12.1 Principal Strains

If the state of stress is given in terms of the principal stress components σ_1, σ_2 and σ_3, the corresponding strains components may be calculated as,

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E} \qquad (5.19)$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} - \nu\frac{\sigma_3}{E} \qquad (5.20)$$

$$\varepsilon_3 = \frac{\sigma_3}{E} - \nu\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} \qquad (5.21)$$

Alternately, we may rewrite as,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \tag{5.22}$$

Using abbreviations, $[\,\varepsilon\,]$ for $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$, $[\,\sigma\,]$ for $\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$ and $[C]$ for $= \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}$

$$[\,\varepsilon\,] = [\,C\,][\,\sigma\,] \tag{5.23}$$

Eq. (5.23) will be useful in writing compact expressions and simplified derivations.

5.12.2 Net Strain Energy Density

Net strain energy density in an element subjected to a general stress field can be obtained by adding (algebraically) the energy density due to all the stress components. That is

$$u = \frac{1}{2}\left[\,\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\,\right] = \frac{1}{2}\sigma^T \varepsilon$$

but $\varepsilon = C\sigma$

$$\therefore\; u = \frac{1}{2}\,\sigma^T C\,\sigma \tag{5.24}$$

Eq. (5.24) may, if desired, be expanded as,

$$u = \frac{1}{2E}\left[\,\sigma_1\,\sigma_2\,\sigma_3\,\right] \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

$$u = \frac{1}{2E}\left[\,\sigma_1\,\sigma_2\,\sigma_3\,\right] \begin{bmatrix} \sigma_1 - \nu\,(\sigma_2 + \sigma_3) \\ \sigma_2 - \nu\,(\sigma_1 + \sigma_3) \\ \sigma_3 - \nu\,(\sigma_1 + \sigma_2) \end{bmatrix}$$

$$u = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\,\sigma_1)\right) \tag{5.25}$$

5.12.3 Components of Strain Energy Density

You may recall that the total stress field on an element may be considered as composed of two component sets, one of which is the dilatation component and the other is the distortion component. Hence, the net strain energy is also divided into two components, namely strain energy of dilatation, u_s and strain energy of distortion, u_d.

If the principal components of stress field are given by σ_1, σ_2 and σ_3, we may write as,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_s \\ \sigma_s \\ \sigma_s \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_s \\ \sigma_2 - \sigma_s \\ \sigma_3 - \sigma_s \end{bmatrix} \tag{5.26}$$

where the dilatation or spherical components of the stress field σ_s is given by

$$\frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \text{ or } \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

Now let us obtain an expression for strain energy of distortion, using the form of Eq. (5.25).

$$u_d = \frac{1}{2E}\left[\,(\sigma_1 - \sigma_s)^2 + (\sigma_2 - \sigma_s)^2 + (\sigma_3 - \sigma_s)^2\,\right]$$

$$-2\nu\left[\,(\sigma_1 - \sigma_s)(\sigma_2 - \sigma_s) + (\sigma_2 - \sigma_s)(\sigma_3 - \sigma_s) + (\sigma_3 - \sigma_s)(\sigma_1 - \sigma_s)\,\right]$$

$$= \frac{1}{18E}\left(\left\{3\sigma_1 - (\sigma_1 + \sigma_2 + \sigma_3)\right\}^2 + \left\{3\sigma_2 - (\sigma_1 + \sigma_2 + \sigma_3)\right\}^2 + \left\{3\sigma_3 - (\sigma_1 + \sigma_2 + \sigma_3)\right\}^2\right)$$

$$-2\nu\left(\left\{3\sigma_1 - (\sigma_1 + \sigma_2 + \sigma_3)\right\}\left\{3\sigma_2 - (\sigma_1 + \sigma_2 + \sigma_3)\right\} + \ldots\right)$$

$$= \frac{1}{18E}\Big((2\sigma_1-\sigma_2-\sigma_3)^2+(2\sigma_2-\sigma_1-\sigma_3)^2+(2\sigma_3-\sigma_1-\sigma_2)^2$$
$$-2v\{(2\sigma_1-\sigma_2-\sigma_3)(2\sigma_2-\sigma_3-\sigma_1)+(2\sigma_2-\sigma_3-\sigma_1)(2\sigma_3-\sigma_1-\sigma_2)+$$
$$(2\sigma_3-\sigma_2-\sigma_1)(2\sigma_1-\sigma_2-\sigma_3)\}\Big)$$

$$= \frac{1}{18E}\Big[6\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)-6(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)+$$
$$6v\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1\right)\Big]$$

$$= \frac{(1+v)}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1\right]$$

$$= \frac{2(1+v)}{6E}\left[\frac{1}{2}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)\right]$$

$$= \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\quad\left(\text{as shear modulus, } G=\frac{E}{2(1+v)}\right)$$

Thus,
$$u_d = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \qquad (5.27)$$

$$u_d = \frac{1}{3G}\left[\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2+\left(\frac{\sigma_3-\sigma_1}{2}\right)^2\right] \qquad (5.28)$$

Eq. (5.28) is more significant since the terms $\frac{\sigma_1-\sigma_2}{2}$, $\frac{\sigma_2-\sigma_3}{2}$ and $\frac{\sigma_3-\sigma_1}{2}$ represent the three extreme values of shear stress components.

The dilatation component of strain energy, u_s is not of much practical significance. However, if desired it may easily be obtained using the form of Eq. (5.25), as

$$u_s = \frac{1}{2E}\left(\sigma_s^2+\sigma_s^2+\sigma_s^2-2v(\sigma_s\sigma_s+\sigma_s\sigma_s+\sigma_s\sigma_s)\right)$$

i.e.
$$u_s = \frac{3}{2E}\left(\sigma_s^2-2v\sigma_s^2\right)$$

or
$$u_s = \frac{3\sigma_s^2(1-2v)}{2E} \qquad (5.29)$$

or
$$u_s = \frac{\sigma_s^2}{2K} \qquad (5.30)$$

where, K is the Bulk Modulus given by $\frac{E}{3(1-2v)}$.

The expressions for strain energy density may be used to obtain, wherever required, to obtain the total strain energy by the expressions

$$U = \int_v u\,dv,\quad U_d = \int_v u_d\,dv \quad\text{and}\quad U_s = \int_v u_s\,dv.$$

Such expressions have wide applications in structural analysis and theory of elasticity. However, in analysing the state of stress, energy density expressions have been found to be adequate.

5.13 CONCEPT OF FAILURE AND EQUIVALENT STRESSES

When we are testing materials for strength, generally, we apply one uniaxial component of stress usually in tension or compression and obtain a value for the limiting state of yield of failure and consider a solid to have failed if that state is reached.

What is meant by reaching the limiting state ? Let us go into the question, a little deeper. When we conduct a tension test on a mild steel rod and find that it yields when the stress reaches a value of 260 MPa, this should be the limiting state as it is obtained by experimentation. But one may put a question whether the material has yielded because the normal stress reached a limit of 260 MPa or on the contrary because the normal strain has reached a value of 0.0013. The question need not stop here. If you calculate the maximum shear stress, you may also ask whether reaching a shear stress of 130 MPa is the limit for yielding of the material. Further question may be "whether there is any other criteria for yield ?"

Every independent answer to this question has led to a failure theory and we will now learn a few of these theories.

5.13.1 Theories of Failure

Different people have prescribed different criteria for failure of a solid and hence, a number of failure theories (also called strength theories) have been formulated.

In what is known as *Principal Stress Theory*, if the maximum principal stress (or minimum principal stress in the case of compression) reaches the same value of maximum principal stress at failure in uniaxial strength test, then the yield or failure limit is considered as reached.

According to *Principal Strain Theory*, a material is considered to have reached the yield or failure limit when the maximum principal strain in the material has reached the value of the maximum principal strain at failure as observed in the uniaxial strength test.

According to *Shear Stress Theory*, a material is considered to have reached the yield or failure limit when the maximum shear stress in the material has reached the value of the maximum shear stress at failure as observed in the uniaxial strength test.

Apart from the stress or strain limits as governing criteria for failure, the capacity of the material to store energy is also considered as criteria for failure and theories have been formulated based on these criteria.

According to *Total Strain Energy Theory*, a material is considered to have reached the yield or failure limit when the total strain energy density (anywhere within the material) has reached the total strain energy density observed at failure in the case of uniaxial strength test.

Yet another failure theory has been formulated, with the assumption that the distortion energy density, rather than the total strain energy density, is significant as failure criteria. According to this theory, known as *Distortion Energy Theory*, a material is considered to have reached the yield or failure limit when the distortion energy density (anywhere within the solid) reaches the value of distortion energy density at failure as observed in the case of uniaxial strength tests.

A few other theories are not so popular and hence, are not dealt with here. Now, you may get a doubt, whether the different theories really set different criteria for failure or are these theories only different ways of expressing the same criteria and hence, are essentially the same ? Let us examine a few simple cases and ascertain the answer.

5.13.2 A Comparison of Different Theories of Failure

We know that mild steel yields at a stress value of 260 MPa. The normal strain at this limit is 0.0013 and maximum shear stress at yield is 130 MPa.

Total strain energy density, [Eq. (5.25)] at yield $u = \frac{\sigma_y^2}{2E} = \frac{260^2}{2E}$

Distortion energy density at yield [Eq. (5.27)] $u_d = \frac{1}{12G}\left(\sigma_y^2 + (-\sigma_y)^2\right)$

$$u_d = \frac{\sigma_y^2}{6G} = \frac{260^2}{6G}$$

Example 5.12

Let us consider a few cases of solids of the same material (mild steel) under different states of stress as shown in Figure 5.12. The given states of stress have already been reduced in terms of principal stresses. Poisson's ratio ν is given as 0.3.

Solution

Case I

First, let us consider the solid shown in Figure 5.12 (a).

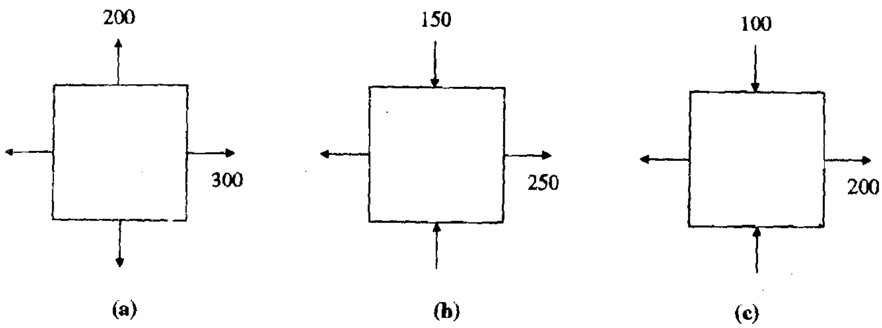

Figure 5.12

(i) Since the major principal stress, σ_1 (300 MPa) is more than σ_y (260 MPa), the solid will fail according to principal stress theory.

(ii) The maximum principal strain, $\varepsilon_1 = \frac{300}{E} - 0.3\,\frac{(+200)}{E}$

$$= \frac{240}{E} < \frac{260}{E}$$

$\therefore$ According to principal strain theory, the solid is safe.

(iii) Maximum shear stress, $\tau_{max} = \frac{300-200}{2} = 50 << 130$

$\therefore$ The solid is safe according to shear stress theory too.

(iv) Total strain energy density in the solid, u is as follows :

$$u = \frac{1}{2E}\left[\,300^2 + 200^2 - 2\times 0.3\times 300\times 200\,\right] = \frac{47000}{E} > \frac{260^2}{2E}$$

$\therefore$ The solid will fail according to total strain energy theory.

(v) Distortion energy density in the solid, u_d as follows :

$$u_d = \frac{1}{12G}\left[\,(300-200)^2 + 200^2 + 300^2\,\right] = \frac{70000}{6G} > \frac{260^2}{6G}$$

$\therefore$ Distortion energy theory also predicts that the solid will fail.

Case II

Let us now consider the solid shown in Figure 5.12 (b).

(i) As the principal stresses are within 260 MPa, the solid is safe according to the principal stress theory.

(ii) The maximum principal strain $\varepsilon_1 = \frac{250}{E} - 0.3\left(\frac{-150}{E}\right) = \frac{295}{E} > \frac{260}{E}$

$\therefore$ The solid will fail according to principal strain theory.

(iii) Maximum shear stress. $\tau_{max} = \frac{250-(-150)}{2} = 200 > 130$

$\therefore$ The solid is not safe as per shear stress theory.

(iv) Total strain energy density $u = \frac{1}{2E}\left[\,250^2 + (-150)^2 - 2\times 0.3\times 250\,(-150)\,\right]$

$$= \frac{53750}{E} > \frac{260^2}{2E}$$

$\therefore$ The solid will fail.

(v) Distortion Energy Density $= \frac{1}{12G}\left(\{250-(-150)\}^2 + 250^2 + 150^2\right)$

$$= \frac{122500}{E} > \frac{260^2}{E}$$

$\therefore$ Distortion energy theory also predicts failure.

Case III

Let us now consider the solid shown in Figure 5.12 (c).

(i) By inspection we may see that the solid is safe according to principal stress theory ($\sigma_1 < 260$).

(ii) Maximum principal strain $\varepsilon_1 = \frac{200}{E} - 0.3\left(\frac{-100}{E}\right) = \frac{230}{E} < \frac{260}{E}$

$\therefore$ The solid is safe according to principal strain theory too.

(iii) Maximum shear stress, $\tau_{max} = \frac{200 - (-100)}{2}$

$= 150 > 130$

$\therefore$ Shear stress theory predicts failure.

(iv) Total strain energy density is as follows :

$$u = \frac{1}{2E}\left[\,200^2 + (-100)^2 - 2 \times 0.3 \times 200\,(-100)\,\right]$$

$$= \frac{31000}{E} < \frac{260^2}{2E}$$

$\therefore$ The solid is safe according to strain energy theory.

(v) Distortion energy density is as follows :

$$u_d = \frac{1}{12G}\left([200 - (-100)]^2 + 200^2 + 100^2\right)$$

$$= \frac{70000}{6G} > \frac{260^2}{6G}$$

$\therefore$ Distortion energy theory predicts failure.

If what is safe according to one theory is unsafe according to another theory while what is safe according to the second theory is unsafe according to yet another theory, you may wonder whether to consider the solid as safe or not. Or should the solid be safe according to all the theories ? That will put a very severe condition for safety. Shall we choose a theory which we like best ? Certainly not, we have to find out which theory correctly represents the failure criteria and choose it. It has been found from experience that the shear stress theory (known as Tresca's Theory) suits brittle materials, while, the distortion energy theory (known as Von Mises' Theory) is suitable for ductile materials. So depending on the nature of the material, the designer can choose the appropriate theory.

You may also note, that it is not enough to be merely safe (irrespective of the theory), but that there should be a sufficient margin of safety, defined by the Factor of Safety chosen depending on the nature of the problem.

5.13.3 Equivalent Stress

Equivalent stress is a concept useful in the design of components undergoing a stress field with multiple components of stress. An equivalent stress corresponding to a given state of stress is the value of uniaxial stress that will produce the same effect (depending on the theory used) as that produced by the given set of stress components.

Example 5.13

Consider the state of stress given in Figure 5.12 (a).

(i) According to principal stress theory the equivalent stress for this case is simply σ_1, i.e. 300 MPa.

(ii) Let us consider principal strain theory.

The principal strain introduced in this case is as follows :

$$\varepsilon_1 = \frac{300}{E} - \left(0.3 \times \frac{200}{E}\right) = \frac{240}{E}$$

This much of strain can be produced in uniaxial tension by a stress of 240 MPa and hence, the equivalent stress according to principal strain theory is 240 MPa.

(iii) Maximum shear stress $\tau_{max} = \dfrac{300-200}{2} = 50$ MPa.

This could be produced by a uniaxial stress of 100 MPa and hence equivalent stress according to shear stress theory is only 100 MPa.

(iv) Strain energy density, $u = \dfrac{1}{2E}\left[\,300^2 + 200^2 - 2\times 0.3\times 300\times 200\,\right]$

$$= \frac{47000}{E} \qquad \text{(i)}$$

If the equivalent uniaxial stress is σ_e, then $u = \dfrac{\sigma_e^2}{2E}$ (ii)

i.e. $\dfrac{\sigma_e^2}{2E} = \dfrac{47000}{E}$

$\therefore$ $\sigma_e = \sqrt{2\times 47000} = 306.6$ MPa

(v) Distortion Energy Density

$$u_d = \frac{1}{12G}\left[\left(300-200\right)^2 + 300^2 + 200^2\right] = \frac{70000}{6G}$$

If equivalent uniaxial stress is σ_e , then

$$u_d = \frac{1}{12G}\left(\sigma_e^2 + \sigma_e^2\right) = \frac{\sigma_e^2}{6G}$$

i.e. $\dfrac{\sigma_e^2}{6G} = \dfrac{70000}{6G}$

$\therefore \sigma_e = \sqrt{70000} = 264.6$ MPa.

Even though we have earlier analysed whether the solid is safe or not according to different theories, only by evaluating the equivalent stress, we are able to get an idea of the margin of safety according to each of the theories.

SAQ 8

Evaluate the equivalent stress values given by different theories of failure for the states of stress given in Figures 5.12 (b) and (c), taking Poisson's Ratio as 0.3.

5.13.4 Factors of Safety and Design

Factor of safety with respect to a state of stress, according to any chosen theory may be defined as the ratio of the yield/failure stress of the material in uniaxial strength test to the equivalent stress according to the theory.

Let us illustrate how the concept is applied in design.

Example 5.14

A mild steel bolt is to be designed to simultaneously carry an axial tensile force of 17 kN along with a shear force of 12 kN. Taking $\sigma_y = 260$ MPa and Poisson's ratio $\nu = 0.32$, find the required diameter of the bolt according to various theories of failure, if the required factor of safety is 2.0.

Solution

Safe stress = $\dfrac{\sigma_y}{\text{Factor of safety}} = \dfrac{260}{2} = 130$ MPa.

Accordingly the equivalent stress should be restricted to 130 MPa.

Let A be the area of cross section of the bolt.

The state of stress will be defined by the components,

$$\sigma_x = \frac{17000}{A}, \quad \sigma_y = 0 \text{ and } \tau_{xy} = \frac{12000}{A}$$

$$\therefore \text{ Principal stresses } \sigma_{1,2} = \frac{17000}{2A} \pm \sqrt{\left(\frac{17000}{2A}\right)^2 + \left(\frac{12000}{A}\right)^2}$$

$$= \frac{1}{A}\,(8500 \pm 14705.44)$$

i.e. $$\sigma_1 = \frac{23205.44}{A} \text{ and } \sigma_2 = -\frac{6205.44}{A}.$$

(i) *Design according to Principal Stress Theory*

Here, $\sigma_1 = 130$, so we have, $\dfrac{23205.44}{A} = 130$

$$\therefore \text{ Area required} = \frac{23205.44}{130} = 178.5034 \text{ mm}^2$$

$$\text{Required diameter of the bolt} = \sqrt{\frac{175.5034 \times 4}{\pi}}$$

$$= 15.708 \text{ mm.}$$

(ii) *Design according to Principal Strain Theory*

According to this theory, we have, $\dfrac{\sigma_1}{E} - \dfrac{\nu\,\sigma_2}{E} = \dfrac{130}{E}$

i.e. $$\frac{23205.44}{AE} - \frac{0.32 \times (-6205.44)}{AE} = \frac{130}{E}$$

$$\frac{25191.181}{AE} = \frac{130}{E}$$

$$\therefore \quad A = \frac{25191.181}{130} = 193.78 \text{ mm}^2.$$

$$\therefore \text{ Required diameter of the bolt} = \frac{4 \times 193.78}{17} = 15.71 \text{ mm}$$

(iii) *Design according to Shear Stress Theory*

According to this theory, we have, $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{130}{2}$

$$\frac{\dfrac{23205.44}{A} - (-\dfrac{6205.44)}{A}}{2} = \frac{130}{2}$$

or $$\frac{29410.88}{2A} = \frac{130}{2}$$

or $$A = \frac{29410.88}{130} = 226.24 \text{ mm}^2.$$

$$\therefore \text{ Required diameter of the bolt} = \sqrt{\frac{4 \times 226.24}{\pi}} = 16.9722 \text{ mm.}$$

(iv) *Design according to Strain Energy Theory*

According to this theory, we have,

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] = \frac{130^2}{2E}$$

$$\sigma_1^2 + \sigma_2^2 - 2\nu\,.\,\sigma_1\,\sigma_2 = 130^2$$

$$\left(\frac{23205.44}{A}\right)^2 + \left(\frac{-6205.44}{A}\right)^2 - 2 \times 0.32 \times \frac{23205.44}{A}\left(\frac{-6205.44}{A}\right) = 130^2$$

i.e. $$\frac{6691599909.2}{A^2} = 130^2$$

$$\therefore \quad A = \left(\frac{6691599909.2}{130^2}\right)^{1/2} = 198.986 \text{ mm}^2$$

$$\therefore \text{ Required diameter of the bolt} = \left(\frac{4 \times 198.986}{\pi}\right)^{1/2} = 15.917 \text{ mm.}$$

(v) *Design according to Distortion Energy Theory*

According to this theory, we have,

$$\frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{130^2}{6G}$$

or $$\frac{(\sigma_1 - \sigma_2)^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_1^2}{2} = 130^2$$

or $$(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 = 2 \times 130^2$$

$$\left(\frac{23205.44}{A} - \frac{(-6205.44)}{A}\right)^2 + \left(\frac{23205.44}{A}\right)^2 + \left(\frac{-6205.44}{A}\right)^2 = 2 \times 130^2$$

i.e. $$\frac{14419997794}{A^2} = 2 \times 130^2$$

$$\therefore \quad A = \left(\frac{14419997794}{2 \times 1302}\right)^{1/2} = 206.8 \text{ mm}^2$$

$\therefore$ Required diameter of the bolt = 16.217 mm.

Note

Whatever be the given state of stress (or even loading) once it is reduced to the state of principal stress components σ_1, σ_2 and σ_3 the member can be designed according to any theory.

SAQ 9

Design the bolt in Example 5.14 by taking σ_y = 250 MPa, Poisson's ratio = 0.333 and Factor of safety = 1.75.

SAQ 10

Design a bolt with the material described in Example 5.14 for a tensile force of 22 kN and a shear force of 14 kN, i.e. with σ_y = 260 MPa, ν = 0.32 and Factor of safety = 2.

SAQ 11

A bolt of 16 mm diameter made of a material with σ_y = 260 MPa and Poisson's ratio = 0.32 is subjected to an axial tensile force of 18 kN and a shear force of 11.6 kN. Evaluate the factor of safety for the bolt according to various theories of failure.

5.14 SUMMARY

This unit is a vital link in the analysis of solids so as to ensure safe design of different components of structures or machines or other systems. Here, you were exposed to a deeper insight into the implications of a given state of stress. You have learnt how to evaluate the stress components on different planes and also to find the extreme values of stress components. In addition, a cursory treatment of the methods of establishing the state of stress in simple cases of combined loading is also provided. This study should be helpful in understanding a given loading situation and its bearing on the strength of solid involved.

Finally, an introductory treatment of different failure theories has been provided with suitable illustrative examples of analysis and design. Rightly, we have come to the close of Block 1 which undertakes a treatment of simple cases of stresses and strains on simple members.

Now, you are armed with the knowledge and skill adequate enough for undertaking a study of more complex systems of structures and developing the capability for designing such systems, an activity which may be considered as backbone of engineering field.

5.15 ANSWERS TO SAQs

SAQ 1

On plane DB, $\sigma_n = 12.31$, $\tau_{nt} = 18.462$ both unsafe; when $\sigma_x = 32.494$, the joint will fail in tension.

SAQ 2

$$\tan 2\psi = \frac{\sigma_x - \sigma_y}{2\,\tau_{xy}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

SAQ 3

(a) $\sigma_{1,2} = 49, -41$

$\phi_{1,2} = 71.565^\circ, -18.435^\circ$

(b) For $\theta = 30^\circ$ $\sigma_n = 9.3827$ and $\tau_{nt} = 44.677$

For $\theta = 45^\circ$ $\sigma_n = 31$ and $\tau_{nt} = 36$

For $\theta = 75^\circ$ $\sigma_n = 48.677$ and $\tau_{nt} = -5.383$

SAQ 4

(a) $\sigma_n = 93.57$ and $\tau_{nt} \leq 42.93$

SAQ 5

(a) $\sigma_{1,2} = 9, 3$ $\phi_{1,2} = -45^\circ, +45^\circ$

$\sigma_{1,2} = \sqrt{45}, -\sqrt{45}$ $\phi_{1,2} = -13.28^\circ, 76.72^\circ$

$\sigma_{1,2} = 19.5, -10.5$ $\phi_{1,2} = 36.65^\circ, -53.35^\circ$

(d) It will be circle of radius 10 with centre at origin.

(e) It will be just the point $(\sigma_x, 0)$.

SAQ 6

$\sigma_{x\,(\max)} = 58.513$,

$\tau_{xy\,(\max)} = 40.953$, and

$\sigma_1 = 100.68$.

SAQ 7

Required diameter $D = 92.67$ mm.

SAQ 8

Case as in Figure	Equivalent Stress (MPa) according to				
	Principal Stress Theory	Principal Strain Theory	Shear Stress Theory	Strain Energy Theory	Distortion Energy Theory
5.12 (b)	250	295	200	327.872	350
5.12 (c)	200	230	150	249	264.6

SAQ 9, 10

Problem	Required Diameter of Bolt in mm according to				
	Principal Stress Theory	Principal Strain Theory	Shear Stress Theory	Strain Energy Theory	Distortion Energy Theory
SAQ 9	14.4	15.01	11.45	14.93	15.47
SAQ 10	16.8	17.42	13.21	17.61	17.91

SAQ 11

Theory	Principal Stress Theory	Principal Strain Theory	Shear Stress Theory	Strain Energy Theory	Distortion Energy Theory
Equivalent Stress	117.78	126.82	73.02	129.62	134.16
Factor of safety	2.2075	2.05	3.56	2.005	1.938

FURTHER READING

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(4) Popov, E. P., (1993), *Mechanics of Materials - Second Edition*, Prentice Hall of India Private Limited.

(5) Ryder, G. H., (1993), *Strength of Materials*, Educational Low Priced Books Scheme.

(6) Case, John, Chilver, L., Ross, Carl T. F., (1993), *Strength of Materials and Structures - Third Edition*, Educational Low Priced Books Scheme.