SAQ 5

A close coiled helical spring is subjected to an axial twist of 14 N m. If the spring has 14 coils with wire diameter of 12 mm and the mean coil diameter of 200 mm, find

(a) the strain energy stored,

(b) maximum bending stress in the wire, and

(c) axial twist.

Take $E = 200$ GPa.

15.3 OPEN COILED HELICAL SPRINGS

In the case of open coiled helical springs, the coils are not so close together, hence, the helix angle α cannot be treated as small.

15.3.1 Spring Subjected to Axial Load

Let W be the axial load acting at the end of the spring as shown in Figure 15.3 (a).

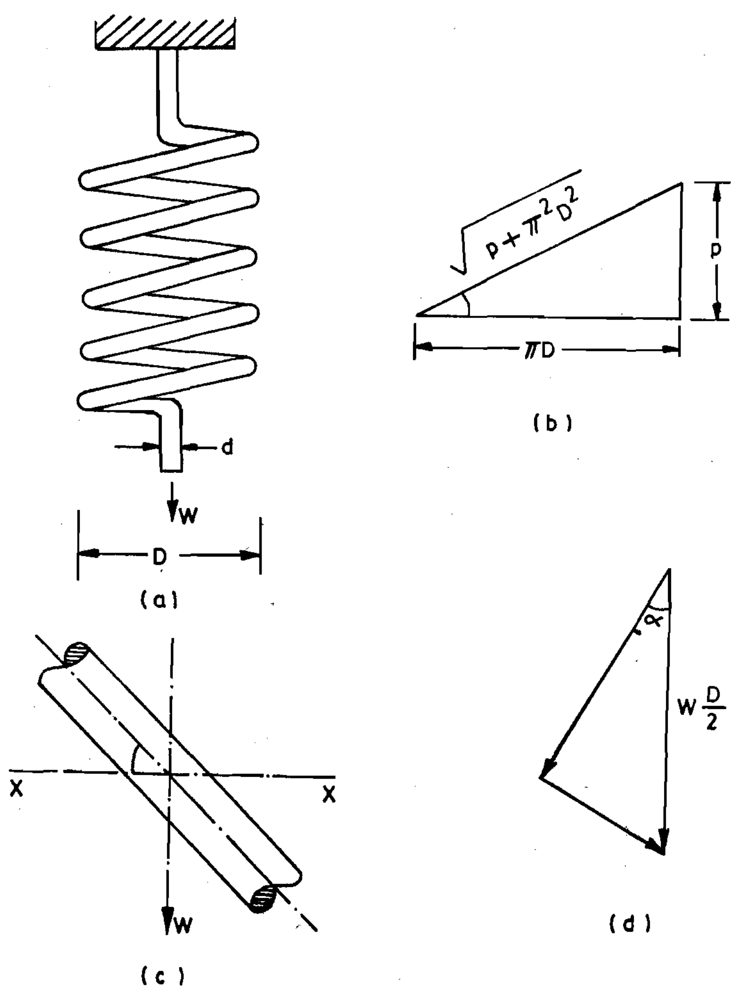

Figure 15.3 : Open Coiled Helical Spring (Subjected to Axial Load)

Let

d = wire diameter,

D = mean coil diameter,

l = length of spring,

n = total number of turns, and

p = pitch of coils.

Moment of any point $M = W \times \left(\frac{D}{2}\right)$

The axial load W exerts a moment $\frac{WD}{2}$ about XX axis.

Resolving along the axis of the spring,

$$\text{bending component} = \frac{WD}{2} \sin \alpha.$$

Resolving normal to the spring axis,

$$\text{twisting of wire} = \frac{WD}{2} \cos \alpha.$$

Let $\quad l$ = Total length of the spring

$$= n\left(\frac{\pi D}{\cos \alpha}\right)$$

$$= n\,\pi D \times \frac{\sqrt{p^2 + \pi^2 D^2}}{\pi D}$$

$$= n\,\sqrt{p^2 + \pi^2 D^2}$$

Total volume of the spring $= \frac{\pi}{4} d^2 \times l = \frac{\pi d^2}{4}\left(n\sqrt{p^2 + n^2 D^2}\right)$

Strain Energy

Strain energy, U = Strain energy due to bending + Strain energy due to torsion

$$U = \frac{(M_b)^2 \times l}{2EI} + \frac{f_s^2}{4G} \text{ (Volume of the spring)} \tag{15.9}$$

$$= \frac{M_b^2 l}{2EI} + \frac{f_s^2}{4G}\left(\frac{\pi}{4} d^2 l\right) \tag{15.9a}$$

But we know, $\quad f_s = \frac{16T}{\pi d^3} \quad$ and $\quad I = \frac{\pi d^4}{64}$

$$\therefore \quad U = \frac{M_b^2 \times l \times 64}{2E \times \pi d^4} + \frac{\left(\frac{16T}{\pi d^3}\right)}{4G} \times \left(\frac{\pi}{4} d^2 l\right)$$

$$= \frac{32\, M_b^2\, l}{\pi d^4 E} + \frac{16\, T^2 l}{\pi d^4 G}$$

$$U = \left(\frac{32\, l}{\pi d^4 E}\right) M_b^2 + \left(\frac{16\, l}{\pi d^4 G}\right) T^2 \tag{15.10}$$

Deflection of Springs

The partial derivative of this total strain energy with respect to the applied load W gives the deflection δ.

Thus, $$\delta = \frac{\partial U}{\partial W} = 2\left(\frac{32\, l}{\pi d^4 E}\right) M_b \frac{\partial M_b}{\partial W} + 2\left(\frac{16 l}{\pi d^4 G}\right) T \frac{\partial T}{\partial W} \tag{15.11}$$

$$M_b = M \sin \alpha = \frac{WD}{2} \sin \alpha$$

$$\therefore \quad \frac{\partial M_b}{\partial M} = \frac{D}{2} \sin \alpha$$

$$T = M \cos \alpha = \frac{WD}{2} \cos \alpha$$

$$\therefore \quad \frac{\partial T}{\partial W} = \frac{D}{2} \cos \alpha$$

Substituting in expression given in Eq. (15.11),

$$\delta = \frac{\partial U}{\partial W} = 2\left(\frac{32\,l}{\pi d^4 E}\right)\frac{WD}{2}\sin\alpha \times \frac{D}{2}\sin\alpha + 2\left(\frac{16 l}{\pi d^4 G}\right)\frac{WD}{2}\cos\alpha \times \frac{D}{2}\cos\alpha \qquad (15.12)$$

Also, $l = n\left(\frac{\pi D}{\cos\alpha}\right)$

$$\therefore\ \delta = \frac{\partial U}{\partial W} = 2\left(\frac{32\,n\pi D}{\pi d^4 E\cos\alpha}\right)\frac{WD^2\sin^2\alpha}{4} + 2\left(\frac{16\,n\pi D}{\pi d^4 G\cos\alpha}\right)\frac{WD^2\cos^2\alpha}{4}$$

$$\therefore\ \delta = \frac{8\,WD^3 n}{d^4\cos\alpha}\left(\frac{2\sin^2\alpha}{E} + \frac{\cos^2\alpha}{G}\right) \qquad (15.13)$$

Stiffness of Springs

Stiffness of spring, $k = \frac{W}{\delta}$

$$\therefore\ k = \left[\frac{d^4\cos\alpha}{8D^3 n\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)}\right] \qquad (15.14)$$

15.3.2 Spring Subjected to Axial Couple

To find the angle ϕ by which the spring is wound up, apply a unit moment along the axis of helix as shown in Figure 15.4.

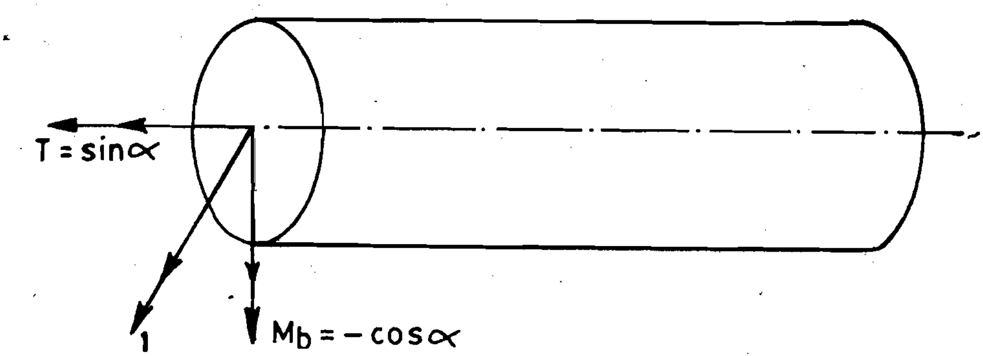

Figure 15.4 : Open Coiled Helical Spring (Subjected to Axial Couple)

Strain Energy

Strain Energy, $U = \left(\frac{32\,l}{\pi d^4 E}\right)M_b^2 + \left(\frac{16\,l}{\pi d^4 G}\right)T^2$

We know, $\phi = \frac{\partial U}{\partial M}$

$$= \left(\frac{32\,l}{\pi d^4 E}\right)2M_b\left(\frac{\partial M_b}{\partial M}\right) + \left(\frac{16\,l}{\pi d^4 G}\right)2T\left(\frac{\partial T}{\partial M}\right)$$

$$= \frac{64\,l\,M_b}{\pi d^4 E}(-\cos\alpha) + \frac{32\,l\,T}{\pi d^4 G}(\sin\alpha)$$

On substituting $M_b = \frac{WD}{2}\sin\alpha$ and $T = \frac{WD}{2}\cos\alpha$,

$$\therefore\ \phi = \frac{16\,WD}{\pi d^4}\,l\sin\alpha\cos\alpha\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16\,WD}{\pi d^4}\left(\frac{n\pi D}{\cos\alpha}\right)\sin\alpha\cos\alpha\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$\therefore\ \phi = \frac{16\,WD^2 n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right) \qquad (15.15)$$

15.3.3 Spring Subjected to Moment along the Axis of the Helix

Figure 15.5 (a) shows an open coiled helical spring subjected to an axial couple M.

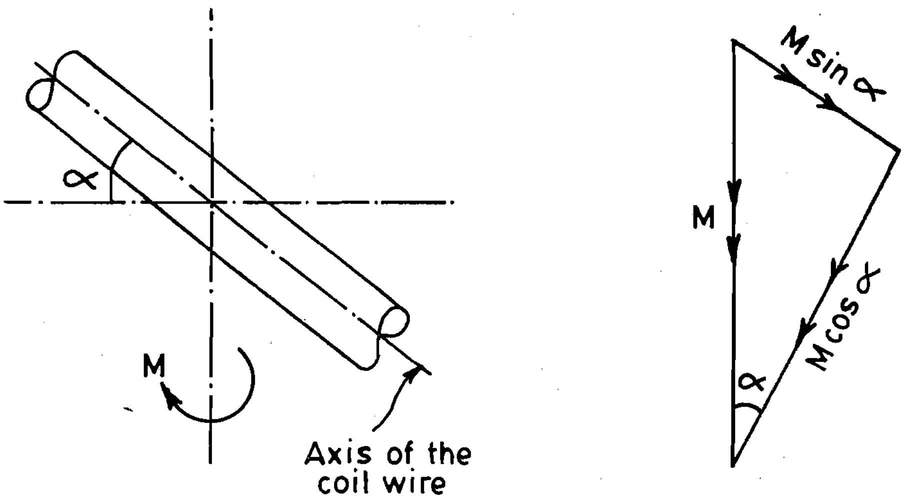

Figure 15.5 : Open Coiled Helical Spring (Subjected to Moment M along the Axis of the Helix)

Refer Figure 15.5 (b), we get

Component along the axis of the coil wire = $M \sin \alpha$ causes twisting of the wire, and

Component at its right angles = $M \cos \alpha$ causes bending of the wire

$$\therefore \; M_b = M \cos \alpha \text{ and } T = M \sin \alpha$$

Strain energy, U = Strain energy due to bending + Strain energy due to torsion

$$= \frac{M_b^2\, l}{2EI} + \frac{f_s^2}{4G} \times \text{(Volume of the spring)}$$

$$= \frac{M_b^2\, l}{2E\left(\frac{\pi d^4}{64}\right)} + \frac{\left(\frac{16T}{\pi d^3}\right)^2}{4G} \times \left(\frac{\pi d^2}{4}\, l\right)$$

$$= \frac{32\, M_b^2\, l}{\pi d^4 E} + \frac{16\, T^2\, l}{\pi d^4 G}$$

$$\therefore \; U = \left(\frac{32\, l}{\pi d^4 E}\right) M_b^2 + \left(\frac{16\, l}{\pi d^4 G}\right) T^2 \qquad (15.16)$$

We know, $$\phi = \frac{\partial U}{\partial M} = \left(\frac{32\, l}{\pi d^4 E}\right) 2 M_b \left(\frac{\partial M_b}{\partial M}\right) + \left(\frac{16\, l}{\pi d^4 G}\right) 2T \left(\frac{\partial T}{\partial M}\right)$$

$$M_b = M \cos \alpha \quad \therefore \; \frac{\partial M_b}{\partial M} = \cos \alpha$$

$$T = M \sin \alpha \quad \therefore \; \frac{\partial T}{\partial M} = \sin \alpha$$

$$\therefore \; \phi = \frac{64\, l}{\pi d^4 E} (M \cos \alpha) \cos \alpha + \frac{32\, l}{\pi d^4 G} (M \sin \alpha) \sin \alpha$$

$$= \frac{32\, Ml}{\pi d^4} \left(\frac{\sin^2 \alpha}{G} + \frac{2 \cos^2 \alpha}{E}\right) \qquad (15.17)$$

On substituting $l = n \dfrac{\pi D}{\cos \alpha}$,

$$\phi = \frac{32\, M}{\pi d^4} \times \frac{n \pi D}{\cos \alpha} \left(\frac{\sin^2 \alpha}{G} + \frac{2 \cos^2 \alpha}{E}\right)$$

$$\therefore \quad \phi = \frac{32\,MnD}{d^4 \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

On putting ($D = 2R$), $$\phi = \frac{64\,MnR}{d^4 \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right) \qquad (15.18)$$

where, R = radius of the coil.

Deflection of Springs

To find the deflection δ of the spring, a unit moment is applied at the end as shown in Figure 15.6.

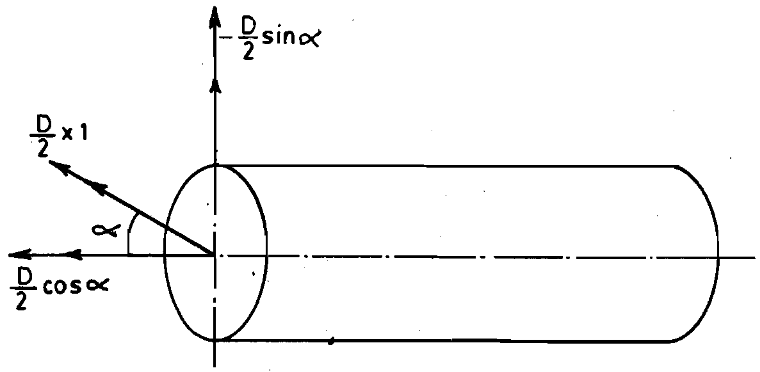

Figure 15.6 : Deflection Calculations

$$\frac{\partial M_b}{\partial W} = -\frac{D}{2}\sin\alpha$$

$$M_b = M\cos\alpha$$

$$\frac{\partial T}{\partial W} = \frac{D}{2}\cos\alpha$$

$$T = \frac{WD}{2}\sin\alpha = M\sin\alpha$$

Strain energy, $$U = \left(\frac{32\,l}{\pi d^4 E}\right)M_b^2 + \left(\frac{16\,l}{\pi d^4 G}\right)T^2$$

$$\frac{\partial U}{\partial W} = \left(\frac{32\,l}{\pi d^4 E}\right)2M_b\left(\frac{\partial M_b}{\partial W}\right) + \left(\frac{16\,l}{\pi d^4 G}\right)2T\left(\frac{\partial T}{\partial W}\right)$$

$$= \left(\frac{32\,l}{\pi d^4 E}\right)2M\cos\alpha\left(-\frac{D}{2}\sin\alpha\right) + \left(\frac{16\,l}{\pi d^4 G}\right)2M\sin\alpha\left(+\frac{D}{2}\cos\alpha\right)$$

$$\frac{\partial U}{\partial W} = \frac{16\,MD\,l\sin\alpha\cos\alpha}{\pi d^4}\left(\frac{1}{G} - \frac{2}{E}\right) \qquad (15.19)$$

Substituting for $l = \dfrac{n\pi D}{\cos\alpha}$

$$\frac{\partial U}{\partial W} = \frac{16\,\pi nD}{\pi d^4 \cos\alpha}\,MD\sin\alpha\cos\alpha\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$\therefore \quad \delta = \frac{\partial U}{\partial W} = \frac{16\,MD^2 n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

15.3.4 Stresses in Springs

(a) Bending stress, $f_b = \dfrac{M_D}{Z}$

where, $M_D = \dfrac{WD}{2}\sin\alpha$ and $Z = \dfrac{\pi d^3}{32}$.

$$\text{Thus,} \therefore \; f_b = \frac{WD}{2} \frac{\sin \alpha}{\left(\frac{\pi d^3}{32}\right)}$$

$$= \frac{16\, WD \sin \alpha}{\pi d^3} \quad (15.21)$$

(b) Shear Stress, $$f_s = \frac{16\, T}{\pi d^3} \quad (15.22)$$

$$= \frac{16 \times \frac{WD}{2} \cos \alpha}{\pi d^3}$$

$$f_s = \frac{8 WD \cos \alpha}{\pi d^3}$$

(c) Principal stresses are as follows :

$$f_{1,2} = \frac{f_b}{2} \pm \sqrt{\frac{f_b^2}{4} + f_s^2}$$

$$= \frac{8\, WD}{\pi d^3} (\sin \alpha \pm 1) \quad (15.23)$$

(d) Maximum shear stress is,

$$(f_s)_{\max} = \left(\frac{f_1 - f_2}{2}\right) = \frac{8\, WD}{\pi d^3} \quad (15.24)$$

15.3.5 Proof Load

As defined earlier, it is the maximum load, the spring can take without getting permanently deformed. For the open coiled helical spring, you can obtain this by equating the maximum allowable stress in shear to the stress equation or by equating the maximum principal stress equation to the allowable stress of the material.

Example 15.6

An open coiled helical spring consisting of 10 turns of 10 mm diameter wire wound to a coil of mean diameter 110 mm. The wire is making an angle of 60° to the axis of the coil which is subjected to an axial load of 90 N. Find the extension of the coil. Take $E = 210$ GPa and $\nu = 0.25$.

Solution

Here, we have $n = 10$ $\quad d = 10$ mm

$D = 110$ mm $\quad W = 90$ N

The wire makes an angle of 60° with the axis of the coil.

$$\therefore \; \alpha = (90 - 60) = 30°$$

For $\alpha = 30^{\circ}$, we get, $\cos \alpha = 0.866$ and $\sin \alpha = 0.500$

We have, $E = 210 \times 10^3$ MPa

$\nu = 0.25$

Using the relationship,

$$E = 2G\,(1 + \nu),$$

We get, $$G = \frac{210 \times 10^3}{2\,(1 + 0.25)} = 84 \times 10^3 \text{ MPa}$$

$$\therefore \text{ Extension, } \delta = \frac{8 W D^3 n}{d^4 \cos \alpha} \left(\frac{\cos^2 \alpha}{G} + \frac{2 \sin^2 \alpha}{E}\right)$$

$$= \frac{8 \times 90 \times (110)^3 \times 10}{(10)^4 \times 0.866} \left(\frac{(0.866)^2}{84 \times 10^3} + \frac{2\,(0.500)^2}{210 \times 10^3}\right)$$

$$\therefore \; \delta = 12.5 \text{ mm.}$$

Example 15.7

An open coiled helical spring of 50 mm mean diameter is made of steel of 6 mm diameter. Calculate the number of turns required in the spring to give a deflection of 12 mm for an axial load of 250 N, if the angle of helix is 30°. Calculate also, the rotation of one end of the spring relative to the other if it is subjected to an axial couple of 10 N m. Take $E = 210$ GPa and $G = 84$ GPa.

Solution

Here, we have, $D = 50$ mm $\quad d = 6$ mm

$\delta = 12$ mm $\quad W = 250$ N

$\alpha = 30°$ $\quad n = ?$

$E = 210$ GPa $\quad G = 84$ GPa

For $\alpha = 30°$, we get, $\sin\alpha = 0.500$, $\cos\alpha = 0.866$

Using the relationship, $$\delta = \frac{8WD^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$12 = \frac{8\times 250\times (50)^3 n}{(6)^4\times 0.866}\left(\frac{(0.866)^2}{84\times 10^3} + \frac{2\times(0.500)^2}{210\times 10^3}\right)$$

from which we get, the number of turns, $n = 4.76$

Here, axial couple $M = 10$ N m $= 10\times 10^3$ N mm

Angle of rotation, $$\phi = \frac{32\,MDn}{d^4\cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32\times(10^4)\,(50)\times 4.76}{(6)^4\,(0.866)}\times\left(\frac{(0.500)^2}{84\times 10^3} + \frac{2\times(0.866)^2}{210\times 10^3}\right)$$

$\therefore\ \phi = 0.6868$ radian.

Example 15.8

An open coiled helical spring consisting of 12 turns of radius 100 mm and diameter of wire 12 mm and the angle of helix 15°. It is subjected to an axial load of 250 N. Determine the deflection under the load and also the angle of rotation of the free end. Take $E = 210$ GPa and $G = 84$ GPa.

Solution

Here, we have, $n = 12$ $\quad R = 100$ mm

$d = 12$ mm $\quad D = 200$ mm

$W = 250$ N

For $\alpha = 15°$, we get $\cos\alpha = 0.9659$, $\sin\alpha = 0.2588$

Given $E = 210\times 10^3$ N/mm^2

$G = 84\times 10^3$ N/mm^2

Using the relationship, $$\delta = \frac{8WD^3n}{d^4\cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8\times 250\times(200)^3\times 12}{(12)^4\times(0.9659)}\left(\frac{(0.9659)^2}{84\times 10^3} + 2\times\frac{(0.2588)^2}{210\times 10^3}\right)$$

$\therefore\ \delta = 112$ mm

Angle of rotation, $$\phi = \frac{16\,WD^2n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16\times 250\times(200)^3\times 12\times(0.2588)}{(12)^4}\left(\frac{1}{84\times 10^3} - \frac{2}{210\times 10^3}\right)$$

$\therefore\ \phi = 0.057$ radian.

Example 15.9

An open coiled helical spring is made of wire of 10 mm diameter. It has 10 coils of mean diameter 50 mm. What is the greatest axial load that can be applied to the ends of the spring if principal stress and maximum shear are not to exceed 100 MPa and 60 MPa respectively. Calculate for this load, the axial and angular deflection of the spring.

Take $E = 200$ GPa, and $G = 80$ GPa.

Solution

Here, we have,

$d = 10$ mm $\quad n = 10$

$D = 50$ mm $\quad R = 25$ mm

$f_1 \leq 100$ MPa $\quad (f_c)_{max} \leq 60$ MPa

$E = 200 \times 10^3$ MPa $\quad G = 80 \times 10^3$ MPa

$$\text{Maximum shear stress, } (f_s)_{max} = \frac{f_1 - f_2}{2} = \frac{8WD}{\pi d^3}$$

$$\therefore \; 60 = \frac{8 \times W \times 50}{\pi (10)^3}$$

$$\therefore \; W = 471.2 \text{ N}$$

Here α is unknown

$$\text{Maximum principal stress, } f_1 = \frac{8WD}{\pi d^3}(\sin\alpha + 1)$$

$$\therefore \; 100 = \frac{8 \times 471.2 \times 50}{\pi (10)^3}(\sin\alpha + 1)$$

$$(\sin\alpha + 1) = 1.667$$

$$\sin\alpha = 0.667$$

$$\cos\alpha = \sqrt{1 - (0.667)^2} = 0.7451$$

$$\text{Axial Deflection, } \delta = \frac{8WD^3 n}{d^4 \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8 \times 471.2 \times (50)^3 \times 10}{(10)^4 \times 0.7451}\left(\frac{(0.7451)^2}{80 \times 10^3} + 2 \times \frac{(0.667)^2}{200 \times 10^3}\right)$$

$$\therefore \; \delta = 0.7 \text{ mm}$$

$$\text{Angular Deflection, } \phi = \frac{16\, WD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16 \times (471.2) \times (50)^2 \times 10 \times 0.667}{(10)^4}\left[\frac{1}{80 \times 10^3} - \frac{2}{200 \times 10^3}\right]$$

$$\therefore \; \phi = 0.0314 \text{ radian.}$$

Example 15.10

An open coiled spring consists of 10 coils, each of mean diameter of 50 mm, the wire forming the coils being 6 mm diameter and making a constant angle of 30° with the planes perpendicular to the axis of the spring. What load would cause the spring to elongate by 12.5 mm and what are the magnitudes of bending and shearing stresses due to this load ? Take $E = 210$ GPa and $G = 84$ GPa.

Solution

Here, we have,

$n = 10 \quad D = 50$ mm

$d = 5$ mm $\quad \delta = 12.5$ mm

$E = 210 \times 10^3$ MPa $\quad G = 84 \times 10^3$ MPa

For $\alpha = 30^\circ$, we get, $\cos\alpha = 0.866$, and $\sin\alpha = 0.5$

Using the relationship, $\delta = \dfrac{8WD^3n}{d^4\cos\alpha}\left(\dfrac{\cos^2\alpha}{G} + \dfrac{2\sin^2\alpha}{E}\right)$

$$12.5 = \frac{8W \times (50)^3 \times 10}{6^4 \times 0.866}\left[\frac{(0.866)^2}{84 \times 10^3} + 2\left(\frac{(0.5)^2}{210 \times 10^3}\right)\right]$$

$$\therefore \quad W = 124.1 \text{ N}$$

Torque, $T = WR\cos\alpha$

Bending moment, $M_b = WR\sin\alpha = M$

Equivalent bending moment,

$$M_{eq} = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$$

$$= \frac{1}{2}\left[WR\sin\alpha + \sqrt{(WR\sin\alpha)^2 + (WR\cos\alpha)^2}\right]$$

$$= \frac{1}{2}(WR\sin\alpha + WR)$$

$$= \frac{WR}{2}(1 + \sin\alpha)$$

$$\therefore \quad M_{eq} = \frac{124.1 \times (25 \times 10^{-3})}{2}(1 + 0.5) = 2.327 \text{ N m}$$

$$\text{Maximum direct stress} = \frac{32\,M_{eq}}{\pi d^3}$$

$$= \frac{32 \times (2.327) \times 10^3}{\pi(6)^3} = 109.7 \text{ N/mm}^2$$

$$\text{Equivalent Torque, } T_{eq} = \sqrt{M^2 + T^2}$$

$$= \sqrt{(WR\sin\alpha)^2 + (WR\cos\alpha)^2}$$

$$= WR$$

$$= 124.1 \times 25 = 3101 \text{ N mm} = 3.101 \text{ N m}$$

$$\text{Maximum direct stress, } f_s = \frac{16\,T_{eq}}{\pi d^3}$$

$$= \frac{16 \times 3101}{\pi(6)^3} = 73.11 \text{ N/mm}^2$$

SAQ 6

In an open coiled helical spring having $\alpha = 20^\circ$, if the inclination of the coils is ignored, calculate the percentage by which the axial extension is underestimated. Take $E = 200$ GPa and $G = 80$ GPa.

SAQ 7

In an open coiled helical spring having $\alpha = 30^\circ$, if the inclination of the coils is neglected, calculate the percentage error in the value obtained for the stiffness. Take $E = 200$ GPa and $G = 80$ GPa.

SAQ 8

An open coiled helical spring having 12 complete turns is made of 15 mm diameter steel rod, the mean diameter of the coil being 100 mm. The angle of helix, α is 15°.

(a) Calculate the deflection under an axial load of 300 N.

(b) Also calculate the direct and shear stresses induced in the section of the wire.

(c) If, however, the axial load of 300 N is replaced by an axial torque of 8 N m, determine the axial deflection and the angle of rotation about the axis of the coil. Take E = 200 GPa and G = 80 GPa.

(d) Also calculate the axial twist which will cause a bending stress of 10 MPa.

SAQ 9

An open coil spring with $\alpha = 30^\circ$ has a vertical displacement of 16.8 mm and an angular rotation of the load end of 0.027 radian under an axial load of 100 N. The spring is formed of a 10 mm diameter steel rod. Calculate the mean radius of the coil Take E = 210 GPa, G = 84 GPa.

SAQ 10

If the close coiled spring formula is used in finding the extension of an open coiled spring under the axial load, determine the maximum angle of helix for which the error in the value of the extension is not to exceed 2 percent. Assume $E = 2.5\ G$.

15.4 COMPOUND SPRINGS

More than one spring may have, at times, to be used to meet the specific requirements. The springs may be used either side by side (in parallel) or connected end to end (in series) as shown in Figures 15.7 (a) and (b). This type of composite system of springs is called as compound springs.

Springs in Parallel

Figure 15.7 (a) shows two springs connected in parallel.

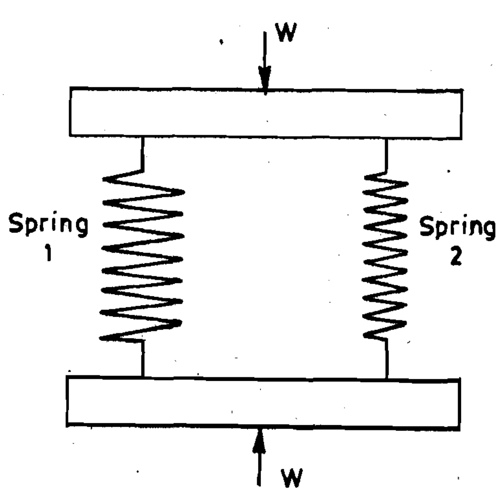

Figure 15.7 (a) : Compound Springs (Spring in Parallel)

In this case,

(a) The extension of both the springs is equal.

(b) The sum of the load carried by each spring is equal to the load W.

$$W = W_1 + W_2 \tag{15.25a}$$

where, W_1 = load shared by the spring 1, and

W_2 = load shared by the spring 2.

Let S_1 and S_2 be the stiffnesses of these two springs respectively.

$$\delta S = \delta S_1 + \delta S_2$$

$$S = S_1 + S_2 \tag{15.25b}$$

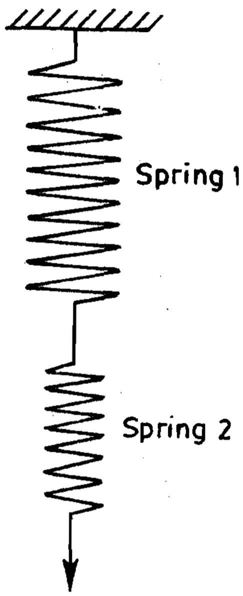

Figure 15.7 (b) Compound Springs (Springs in Series)

Springs in Series

Figure 15.7 (b) shows two springs connected in series.

Each spring carries the same load W applied at the end and the total deflection is equal to the sum of the deflection in each spring.

$$\therefore \ \delta = \delta_1 + \delta_2 \tag{15.26a}$$

But, stiffness of the spring, $S = \dfrac{W}{\delta}$

$$\therefore \ \frac{W}{S} = \frac{W}{S_1} + \frac{W}{S_2}$$

or

$$\frac{1}{S} = \left(\frac{1}{S_1} + \frac{1}{S_2}\right) \tag{15.26b}$$

Example 15.11

A composite spring has two close coiled helical springs connected in series, each spring has 12 coils at a mean diameter of 25 mm. Find the diameter of the wire in one of the springs if the diameter of the wire in the other spring is 2.5 mm and stiffness of the composite spring is 700 N/m. Estimate the greatest load that can be carried by the composite spring for a maximum shearing stress of 180 MPa. Take G = 80 GPa.

Solution

Here, we have $n_1 = 12$ $\quad n_2 = 12$

$D_1 = 25$ mm $\quad D_2 = 25$ mm

Springs are connected in series.

$d_1 = 2.5$ mm $\quad d_2 = ?$

For composite spring,

$$S = 700 \text{ N/m} = 0.7 \text{ N/mm}$$

$$G = 80 \text{ GPa} = 80 \times 10^3 \text{ N/mm}^2$$

Using the relationship, $S = \dfrac{W}{\delta}$

$$\therefore \ \frac{1}{\delta} = \left(\frac{1}{0.7}\right)$$

$$\delta = \frac{64\, WR^3 n}{Gd^4}$$

$$\therefore \ \frac{1}{S} = \frac{\delta}{W} = \frac{64\, R^3 n}{Gd^4}$$

For spring 1,

$$\frac{\delta_1}{W} = \frac{1}{S_1} = \frac{64\, R_1^3 n_1}{Gd_1^4}$$

$$= \frac{64 \times (12.5)^3 \times 12}{80 \times 10^3 \times (2.5)^4}$$

For spring 2, $\frac{\delta_2}{W} = \frac{1}{S_2} = \frac{64 \times (12.5)^3 \times 12}{(80 \times 10^3)\,(d_2)^4}$

For springs connected in series,

$$\frac{1}{S} = \frac{1}{S_1} + \frac{1}{S_2}$$

$$\frac{1}{0.7} = \frac{64 \times (12.5)^3 \times 12}{(80 \times 10^3)\,(2.5)^4} + \frac{64 \times (12.5)^3 \times 12}{(80 \times 10^3)\,(d_2)^4}$$

$$= \frac{64 \times 12.5^3 \times 12}{80 \times 10^3} \left(\frac{1}{(2.5)^4} + \frac{1}{(d_2)^4} \right)$$

$$\left(\frac{1}{(2.5)^4} + \frac{1}{(d_2)^4} \right) = 0.07619$$

$$\frac{1}{(d_2)^4} = 0.05058$$

$$\therefore\ d_2 = 2.109 \text{ mm}$$

Given $(f_s)_{max} = 180$ MPa, then $W = ?$

$$f_s = \frac{8WD}{\pi d^3}$$

$$\therefore\ W = \frac{f_s \times \pi d^3}{8D}$$

Since both the springs carry the same load W,

For spring 1, $W = \frac{180 \times \pi\,(2.5)^3}{8 \times 25} = 44.17$ N

For spring 2, $W = \frac{180 \times \pi\,(2.109)^3}{8 \times 25} = 26.51$ N

Thus, the greatest load the spring can carry based on the maximum shear stress criteria is lesser of these two, i.e. 26.51 N.

Example 15.12

A rigid bar weighing 5 kN and carrying a load of 20 kN is supported by 3 springs as shown in Figure 15.8, having spring constants $S_1 = 30$ kN/m, $S_2 = 18$ kN/m and $S_3 = 12$ kN/m. If the unloaded springs are all of the same length, find the distance x such that the bar is horizontal.

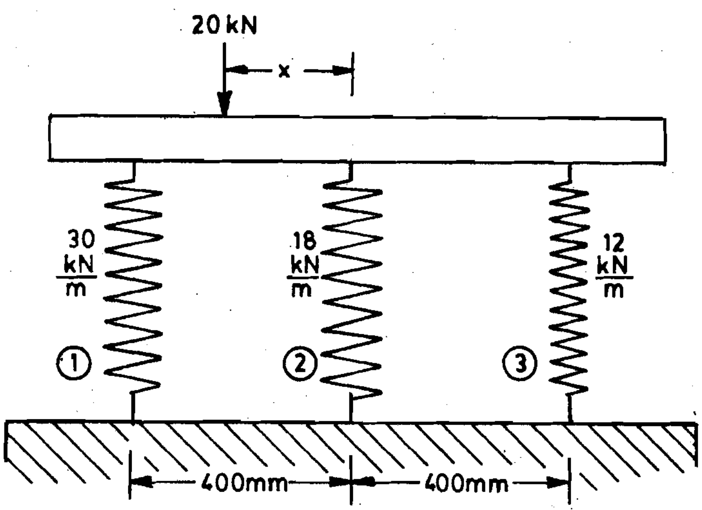

Figure 15.8 : Figure for Example 15.12

Solution

Here the springs are connected in parallel.

We know, spring constant, $S = \left(\frac{W}{\delta}\right)$

Using the relationship, $\delta = \frac{8WD^3 n}{Gd^4}$

We get, $S = \frac{W}{\delta} = \frac{Gd^4}{8D^3 n}$

Here, Deflection of all the 3 springs will be equal, i.e. $\delta = \frac{W}{S}$ = constant.

$$\therefore \quad \frac{W_1}{S_1} = \frac{W_2}{S_2} = \frac{W_3}{S_3}$$

where, W_1, W_2 and W_3 are the loads carried by these 3 springs and S_1, S_2 and S_3 are their respective stiffnesses.

$$S_1 = 30 \text{ kN/m}$$

$$S_2 = 18 \text{ kN/m}$$

$$S_3 = 12 \text{ kN/m}$$

$$W = W_1 + W_2 + W_3 = 20 + 5 = 25 \text{ kN} \qquad \text{(i)}$$

$$\therefore \quad \frac{W_1}{30} = \frac{W_2}{18} = \frac{W_3}{12} \qquad \text{(ii)}$$

$$W_2 = 0.6\, W_1$$

$$W_3 = 0.4\, W_1$$

Substituting in (i), $\quad W_1 + W_2 + W_3 = 25$ kN

$$W_1 + 0.6\, W_1 + 0.4\, W_1 = 25$$

$$W_1 = 12.5 \text{ kN}$$

$$W_2 = 7.5 \text{ kN}$$

$$W_3 = 5.0 \text{ kN}$$

Taking moments about the spring 3,

$$(W_1 \times 0.8) + (W_2 \times 0.4) = (5 \times 0.4) + 20\,(0.4 + x)$$

$$(12.5 \times 0.8) + (7.5 \times 0.4) = 2 + 20\,(0.4 + x)$$

$$\therefore \; x = 0.150 \text{ m}$$

The load of 20 kN must be placed 150 mm from the middle spring.

SAQ 11

Two coil springs whose properties are given below are connected in series :

Spring $A \rightarrow$ coil diameter : 60 mm ; number of coils : 10

Spring $B \rightarrow$ coil diameter : 50 mm ; number of coils : 8

If the wire diameter of spring A be 8 mm and the stiffness of the compound spring be 10 kN/m, determine the wire diameter of spring B. What could be the safe load for the spring so that the shear stress in the wire does not exceed 100 MPa.
Take G = 80 GPa.

SAQ 12

The following data applies to two close coiled helical springs :

Spring	n	d	D	Axial length uncompressed
A	8	6 mm	100 mm	70 mm
B	10	5 mm	80 mm	80 mm

Spring B is placed inside spring A and both are compressed between a pair of parallel plates until the distance between the plates measures 60 mm.

Calculate

(a) the load applied to the plates, and

(b) the maximum intensity of shear stress in each spring.

Take $G = 80$ GPa.

15.5 LEAF SPRINGS

This type of springs are commonly used in carriages such as cars, railway wagons etc., and they are also termed as laminated or carriage springs. It is made up of a number of leaves of equal width and thickness, but varying length placed in laminations and loaded as a beam. The lengths of the plates are so adjusted that the maximum bending stress remains same in every plate and thereby it behaves like a beam of uniform strength.

It is assumed that each plate is free to slide relative to the adjacent plates as the spring deflects and the ends of each plate are tapered to provide a uniform change in effective breadth. Further it is assumed that the plates are bent to the same radius so that they contact only at their edges.

15.5.1 Stress in Springs

Figure 15.9 shows a carriage spring carrying a central vertical load W, which is balanced by equal end reactions $\frac{W}{2}$.

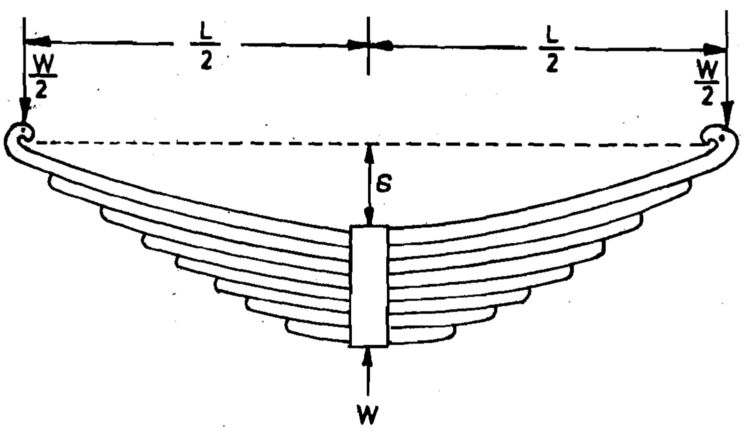

Figure 15.9 : Leaf Spring

Let W = load on the spring

R = initial radius of curvature of plates

δ = initial central deflection

b = width of each plate

t = thickness of each plate

n = number of leaves (plates)

L = span of the spring

δ = bending stress

Section modulus for a single plate or laminate = $\frac{bt^2}{6}$.

Section modulus for the whole spring (having n laminates) = $n\left(\frac{bt^2}{6}\right)$

Maximum bending moment, $M = \frac{WL}{4}$

But $\frac{M}{I} = \frac{f}{y}$

or $M = fZ$

$$\therefore \frac{WL}{4} = f\left(\frac{n\,bt^2}{6}\right)$$

$$f = \frac{3}{2}\times\frac{WL}{nbt^2}$$

Strain Energy

Resilience due to bending = $\frac{f^2}{6E}$

$\therefore$ Total strain energy, $U = \frac{f^2}{6E}\times$ (Volume of the equivalent plate)

Volume = $\left(\frac{nb}{2}Lt\right)$

$\therefore$ Strain Energy $U = \frac{f^2}{6E}\left(\frac{nb}{2}Lt\right)$

Substituting for f_1,

$$U = \frac{\left(\frac{3}{2}\times\frac{WL}{nbt^2}\right)^2}{6E}\times\left(\frac{nb}{2}Lt\right)$$

Work done by the load = $\frac{1}{2}\times W\times\delta$

Equating these two,

$$\frac{1}{2}(W\times\delta) = \frac{\left(\frac{3}{2}\times\frac{WL}{nbt^2}\right)^2}{12E}\,nbLt$$

$$\therefore \delta = \frac{3WL^3}{8\,Enbt^3}$$

Stiffness of Springs

It is defined as the load required to produce unit deflection.

$\therefore$ Spring constant, $S = \frac{W}{\delta}$

$$= \frac{8\,Enbt^3}{3L^3}$$

Proof Load

If W_0 is the load required to make the spring flat, it is known as the proof load.

If δ_0 is the deflection corresponding to proof load w_0,

then, $$\delta_0 = \frac{3W_0L^3}{8Enbt^3}$$

$$\therefore \quad \text{Proof Load,} \quad W_0 = \frac{8Enbt^3}{3L^3}\delta_0$$

15.5.2 Practical Applications

Leaf springs are extensively used in railway carriages, railway wagons, trucks, trollies, buses and cars etc. the common purpose of all kinds of springs is to absorb energy and to release it as and when required. Carriage springs are used normally to absorb shock. In other words, they act as primarily shock absorbers.

Example 15.13

A leaf spring 0.8 m long consists of 12 plates, each of them is 65 mm wide and 6 mm thick. It is simply supported at its ends. The greatest bending stress is not to exceed 180 MPa and the central deflection when the spring is fully loaded is not to exceed 20 mm. Estimate the magnitude of the greatest central load that can be applied to the spring. Take $E = 200 \times 10^3$ MPa.

Solution

Here, we have, $L = 0.8$ m $\quad n = 12$

$b = 65$ mm $\quad t = 6$ mm

$f \leq 210$ N/mm^2 $\quad \delta \leq 20$ mm

Using the relationship, $$f_{max} = \frac{3WL}{2nbt^2}$$

$$180 = \frac{3W \times 800}{2 \times 12 \times 65 \times (6)^2}$$

$$\therefore \quad W = 4212 \text{ N}$$

Using the relationship, $$\delta = \frac{3}{8} \times \frac{WL^3}{Enbt^3}$$

$$20 = \frac{3}{8} \times \frac{W \times (800)^3}{(200 \times 10^3) \times 12 \times (65) \times (6)^3}$$

$$\therefore \quad W = 3510 \text{ N}$$

Thus, the greatest central load that can be applied is lesser of these two, i.e. 3.51 kN.

Example 15.14

A leaf spring is required to satisfy the following specification :

$L = 0.75$ m, $W = 5$ kN, $b = 75$ mm, maximum stress = 210 MPa,

Maximum deflection = 25 mm, $E = 200$ GPa.

Find the number of leaves, their thicknesses and initial radius of curvature.

Solution

Here, we have, $L = 0.75$ m $= 750$ mm $\quad W = 5$ kN $= 5000$ N

$\delta \leq 25$ mm $\quad f \leq 210$ N/mm^2

Maximum stress, $$f = \frac{3}{2} \times \frac{WL}{nbt^2}$$

$$210 = \frac{3}{2} \times \frac{5000 \times 750}{75 \times nt^2}$$

$$\therefore \quad nt^2 = 357.2 \qquad \text{(i)}$$

Maximum deflection, $\delta = \frac{3}{8} \times \frac{WL^3}{Enbt^3}$

$$25 = \frac{3}{8} \times \frac{5000 \times 750^3}{200 \times 10^3 \times (75) \times nt^3}$$

$$\therefore \; nt^3 = 2109 \qquad \text{(ii)}$$

From (i) and (ii), $t = 5.905$ mm

Thus, use 6 mm thick plates.

$$\therefore \; n = \frac{357.2}{6^2} = 9.916$$

Adopt 10 leaves.

Using property of circle, $\frac{L}{2} \times \frac{L}{2} = \delta \, (2R - \delta)$

$$\frac{L^2}{4} = 2R\delta - \delta^2$$

On account of δ^2 being very small, can be neglected.

$$\therefore \; \frac{L^2}{4} = 2R\delta$$

or
$$\delta = \frac{L^2}{8R}$$

and
$$R = \frac{L^2}{8\delta} = \frac{750^2}{8 \times 25} = 2815 \text{ mm}$$

$\therefore$ Radius of curvature = 2.815 m.

SAQ 13

A steel carriage spring is 600 mm long and carries a central load of 4.5 kN. Each plate is 75 mm wide and 6 mm thick. The stress is not to exceed 170 MPa. Calculate the required number of plates and 152 deflection at the centre of the spring. Take E = 210 GPa.

SAQ 14

A laminated spring having a length of 800 mm is required to carry a central point load of 8 kN. Calculate the thickness, width and the number of plates if the bending stress and central deflection are not to exceed 200 N/mm^2 and 20 mm respectively. Also calculate the radius to which the plates should be curved. Take E = 200 GPa. Assume the width of the plate to be 12 times its thickness.

15.6 SUMMARY

We conclude by summarising what we have covered in this unit. We have

(a) Studied the definition of proof load, spring constant and proof stress.

(b) The defferent types of springs namely close-coiled, open-coiled helical springs and leaf springs.

(c) Obtained expressions for stresses in the springs, stiffness of springs and deflection of springs for the above three types of springs.

(d) Also seen the compound springs i.e. springs in series and springs in parallel and the situations where they are used.

(e) Studied some of the practical applications where springs are very often used.

15.7 ANSWERS TO SAQs

SAQ 1

$W = 351.8$ N, $\delta = 77.2$ mm, $U = 13.59$ N m, $k = 4.56$ N/mm.

SAQ 2

$d = 45.92$ mm, $D = 459.22$ mm, $f_s = 29.3$ N/mm^2.

SAQ 3

$D = 63.81$ mm, $n = 6.73$.

SAQ 5

$U = 4.23$ N m, $f_{max} = 82.52$ MPa, $\phi = 34.7°$.

SAQ 6

3.78%.

SAQ 7

9.7%.

SAQ 8

(a) 7.26 mm.

(b) 28.49, –16.78, 22.64 MPa.

(c) 0.1963 mm, 3.659°.

(d) 3.43 N m.

SAQ 9

56.7 mm.

SAQ 10

14.67°.

SAQ 11

6.8 mm, 247 N.

SAQ 12

40.6 N, 19.1 MPa, 39.76 MPa.

SAQ 13

8.82, 11.9 mm.

SAQ 14

$t = 8$ mm, $b = 96$ mm, $n = 7.81$, $R = 4$ m.

FURTHER READING

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