Lagrange's Linear equation.

The equation of the form $Pp + Qq = R$ where P, Q and R are functions of x, y, z is known as Lagrange's linear equation.

To solve this equation, it is enough to solve the subsidary (or) auxillary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The auxillary eq can be solved in two ways

i) method of grouping

ii) method of multipliers.

Method of grouping:

In the auxillary eq. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

If the variables can be separated in any pair of eqs, then we get a soln of the form

$u(x,y) = c_1$ and $v(x,y) = c_2$

(ie) $\phi(u,v) = 0$, ϕ is arbitrary.

(ii) $\phi(c_1, c_2) = 0$

I. $\int \frac{dx}{P} = \int \frac{dy}{Q}$, II. $\int \frac{dy}{Q} = \int \frac{dz}{R}$. III. $\int \frac{dz}{R} = \int \frac{dx}{P}$

1. Solve $x^2 p + y^2 q = z^2$

The eq is of the form $Pp + Qq = R$

where $P = x^2$, $Q = y^2$, $R = z^2$

The auxillary eq is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

Take $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

$\Rightarrow \int x^{-2} dx = \int y^{-2} dy$

$\Rightarrow \frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$

$\Rightarrow -\frac{1}{x} = \frac{-1}{y} + c_1$

$\Rightarrow c_1 = \frac{-1}{x} + \frac{1}{y}$

$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$

$\Rightarrow \int y^{-2} dy = \int z^{-2} dz$

$\Rightarrow \frac{y^{-1}}{-1} = \frac{z^{-1}}{-1} + c_2$

$\Rightarrow -\frac{1}{y} = \frac{-1}{z} + c_2$

$\Rightarrow c_2 = -\frac{1}{y} + \frac{1}{z}$

$\therefore$ The soln is $\phi(c_1, c_2) = 0$

$\Rightarrow \phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$

2. Solve $\frac{y^2 z}{x} p + xzq = y^2$

$P = \frac{y^2 z}{x}$, $Q = xz$, $R = y^2$

The auxillary eq is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$

Take $\int \frac{dx}{\frac{y^2 z}{x}} = \int \frac{dy}{xz}$

$\int x^2 dx = \int y^2 dy$

$\int \frac{dx}{\frac{y^2 z}{x}} = \int \frac{dz}{y^2}$

$\int x dx = \int z dz$

$$\frac{x^3}{3} = \frac{y^3}{3} + c_1$$

$$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = c_1$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = c_2$$

$$\therefore \phi(c_1, c_2) = 0$$

$$\Rightarrow \phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{x^2}{2} - \frac{z^2}{2}\right) = 0$$

$$\Rightarrow \phi(x^3 - y^3, x^2 - z^2) = 0.$$

(M) Method of Multipliers

3. Solve $px + qy = x$

$$Pp + Qq = R \Rightarrow P = x, Q = y, R = x$$

The AE is $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$

$$\frac{dx}{x} = \frac{dy}{y}$$

Take $\int \frac{dx}{x} = \int \frac{dy}{y}$

$$\log x = \log y + \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \Rightarrow$$

$$\frac{dx}{x} = \frac{dz}{x}$$

$$\int \frac{dx}{x} = \int \frac{dz}{x}$$

$$x = z + c_2$$

$$x - z = c_2.$$

$$\therefore \phi(c_1, c_2) = 0$$

$$\Rightarrow \phi\left(\frac{x}{y}, x - z\right) = 0$$

4. Solve $p\tan x + q\tan y = \tan z$

$$Pp + Qq = R \Rightarrow P = \tan x, Q = \tan y, R = \tan z$$

The A.E is $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\int \cot x\, dx = \int \cot y\, dy$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \cot y\, dy = \int \cot z\, dz$$

$\log(\sin x) = \log(\sin y) + \log c_1$

$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$

$c_1 = \frac{\sin x}{\sin y}$

$\log(\sin y) = \log(\sin z) + \log c_2$

$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$

$c_2 = \frac{\sin y}{\sin z}$

$\phi(c_1, c_2) = 0.$

$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$

5. Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

$Pp + Qq = R \Rightarrow P = \sqrt{x}, \; Q = \sqrt{y}, \; R = \sqrt{z}$

The A.E is $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$

$2\sqrt{x} = 2\sqrt{y} + 2c_1$

$\sqrt{x} - \sqrt{y} = c_1$

$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$

$2\sqrt{y} = 2\sqrt{z} + 2c_2$

$\sqrt{y} - \sqrt{z} = c_2$

$\phi(c_1, c_2) = 0$

$\Rightarrow \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$