Definition:

A partial differential equation is an equation involving a function of 2 or more variables and sum of its partial derivatives.

Notation: $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Formation of partial differential equation:

i) Eliminating arbitary constants

ii) Eliminating arbitary functions.

* Eliminating arbitary constant:

Type 1: Number of arbitary contants $\leq$ No. of Independent Variables then we get the I order PDE.

1. Form the PDE by eliminating Arbitary constant from $z = ax + by + a^2 + ab + b^2$

Diff p.w.r.t 'x' | Diff p.w.r.t 'y'

$\frac{\partial z}{\partial x} = a \Rightarrow p = a.$ | $\frac{\partial z}{\partial y} = b \Rightarrow q = b$

$$z = px + qy + p^2 + pq + q^2$$

2. Form the PDE by E.A.C from

$z = (x-a)^2 + (y-b)^2 + 1$

Diff p.w.r.t 'x' | Diff p.w.r.t 'y'

$\frac{\partial z}{\partial x} = 2(x-a) \Rightarrow \frac{p}{2} = x - a$ | $\frac{\partial z}{\partial y} = 2(y-b) \Rightarrow \frac{q}{2} = y - b.$

$$\Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1$$

$$z = \frac{p^2}{4} + \frac{q^2}{4} + 1$$

3. Form the PDE by E.A.C from $\log(az-1) = x+ay+b$

diff p.w.r.t 'x'

$$\frac{1}{az-1}\, a\frac{\partial z}{\partial x} = 1$$

$$\frac{1}{az-1}\, ap = 1 \quad -①$$

diff p.w.r.t 'y'

$$\frac{1}{az-1}\, a\frac{\partial z}{\partial y} = a$$

$$\frac{1}{az-1}\, q = 1 \quad -②$$

$$\frac{1}{az-1}\, ap = \frac{1}{az-1}\, q$$

$$a = \frac{q}{p} \quad -③$$

from ② $\Rightarrow$ $q = az-1 \Rightarrow q+1 = az. \Rightarrow a = \frac{q+1}{z}$ -④

$$\frac{q}{p} = \frac{q+1}{z}$$

$$(q+1)\,p = zq.$$

$$pq+p = zq.$$

$$zq - pq - p = 0. \Rightarrow pq + p - qz = 0$$

Type 2: No. of arbitary constants > No. of independent variable then we get 2nd order PDE.

1. Form the PDE by E.A.C from $z = ax+by+cxy$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = a + cy \quad -①$$

$$p - cy = a$$

diff ① p.w.r.t 'x'

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$r = 0$$

diff p.w.r.t 'y'

$$\frac{\partial z}{\partial y} = b + cx \quad -②$$

$$q - cx = b$$

diff ② p.w.r.t 'y'

$$\frac{\partial^2 z}{\partial y^2} = 0$$

$$t = 0$$

diff ① p.w.r.t 'y'

$$\frac{\partial^2 z}{\partial x\,\partial y} = c \Rightarrow s = c$$

$\Rightarrow z = (p - cy)x + (q - cx)y + sxy$

$= px - cyx + qy - cxy + sxy$

$= px + qy - 2cxy + sxy.$

$\Rightarrow z = px + qy - 2sxy + sxy$

$\Rightarrow z = px + qy - sxy.$

Eliminating Arbitrary Function

Type 1:

Eliminating one arbitrary function then we get 1st order P.D.E

1) Form the PDE by E.A.F from $z = f(x^2 + y^2 + z^2)$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$p = f'(x^2+y^2+z^2)\,2(x+zp) \quad -①$$

diff p.w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$q = f'(x^2+y^2+z^2)\,2(y+zq) \quad -②$$

$\div$ ① by ②.

$$\frac{p}{q} = \frac{f'(x^2+y^2+z^2)\,2(x+zp)}{f'(x^2+y^2+z^2)\,2(y+zq)}$$

$$\frac{p}{q} = \frac{x+zp}{y+zq}$$

$$p(y+zq) = q(x+zp)$$

$$py + zpq = qx + zpq$$

$$py = qx.$$

2) $z = xyz + \phi(x^2+y^2-z^2)$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = \left[yz + xy\frac{\partial z}{\partial x}\right] + \phi'(x^2+y^2-z^2)\left(2x - 2z\frac{\partial z}{\partial x}\right)$$

$$p - yz - xyp = \phi'(x^2+y^2-z^2)\,2(x-zp) \quad \text{— (1)}$$

diff p.w.r.t 'y'

$$\frac{\partial z}{\partial y} = \left[xy\frac{\partial z}{\partial y} + xz\right] + \phi'(x^2+y^2-z^2)\left(2y - 2z\frac{\partial z}{\partial y}\right)$$

$$q - xz - xyq = \phi'(x^2+y^2-z^2)\,2(y-zq) \quad \text{— (2)}$$

$\div$ (1) by (2)

$$\frac{p-yz-xyp}{q-xz-xyq} = \frac{\cancel{\phi'(x^2+y^2-z^2)}\,2(x-zp)}{\cancel{\phi'(x^2+y^2-z^2)}\,2(y-zq)}$$

$$\frac{p-yz-xyp}{q-xz-xyq} = \frac{x-zp}{y-zq}$$

$$(y-zq)(p-yz-xyp) = (x-zp)(q-xz-xyq)$$

$$yp - y^2z - xy^2p - \cancel{zpq} + yqz^2 + \cancel{xyzpq} = xq - x^2z - x^2yq - \cancel{zpq} + xz^2p + \cancel{xyzpq}$$

$$py - y^2z - xy^2p + z^2yq - qx + x^2z + x^2yq - z^2xp = 0$$

Type 2:

Eliminating two arbitrary function then we get 2nd order P.D.E.

1) Eliminating arbitrary function from $z = f(ax+by) + g(cx+dy)$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = af'(ax+by) + cg'(cx+dy)$$

$$p = af'(ax+by) + c\,g'(cx+dy) \quad \text{— (1)}$$

diff p.w.r.t 'y'

$$\frac{\partial z}{\partial y} = bf'(ax+by) + dg'(cx+dy).$$

$$q = bf'(ax+by) + dg'(cx+dy) \quad -②$$

diff ① p.w.r.t 'x' & diff ② p.w.r.t 'y'

$$r = \frac{\partial^2 z}{\partial x^2} = a^2 f''(ax+by) + c^2 g''(cx+dy) \quad -③$$

$$t = \frac{\partial^2 z}{\partial y^2} = b^2 f''(ax+by) + d^2 g''(cx+dy) \quad -④$$

diff ① p.w.r.t 'y'

$$s = \frac{\partial^2 z}{\partial x \partial y} = ab f''(ax+by) + cd g''(cx+dy) \quad -⑤$$

Eliminating ~~constants~~ functions b/w ③, ④ & ⑤

$$\begin{vmatrix} r & a^2 & c^2 \\ s & ab & cd \\ t & b^2 & d^2 \end{vmatrix} = 0$$

$$\Rightarrow r[abd^2 - b^2cd] - a^2[sd^2 - tcd] + c^2[sb^2 - tab] = 0.$$

$$\Rightarrow abd^2r - b^2cdr - a^2sd^2 + a^2ctd + b^2c^2s - abc^2t = 0$$

Type 3:

Eliminating of arbitrary functions from $f(u,v) = 0$

$u, v \to$ functions of x and y

$$f(u,v) = 0.$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

1. Form the PDE by E.A.F from $\phi(x^2+y^2+z^2, ax+by+cz) = 0$

$$u = x^2+y^2+z^2$$

$$\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$$

$$= 2x + 2zp$$

$$V = ax+by+cz$$

$$\frac{\partial v}{\partial x} = a + c\frac{\partial z}{\partial x}$$

$$= a + cp$$

$$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y}$$

$$= 2y + 2zq$$

$$\frac{\partial v}{\partial y} = b + c\frac{\partial z}{\partial y}$$

$$= b + cq$$

$$f(u,v) = 0$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2x + 2zp & a + cp \\ 2y + 2zq & b + cq \end{vmatrix} = 0$$

$$\Rightarrow (2x + 2zp)(b + cq) - (a + cp)(2y + 2zq) = 0$$

$$\Rightarrow 2bx + 2cxq + 2bzp + \cancel{2zpcq} - 2ay - 2azq - 2cpy - \cancel{2cpzq} = 0$$

$$\Rightarrow bx - ay + cxq - cpy + bzp - azq = 0$$

2. $f(xy + z^2, x + y + z) = 0$

$$u = xy + z^2$$

$$\frac{\partial u}{\partial x} = y + 2z\frac{\partial z}{\partial x}$$

$$= y + 2zp$$

$$\frac{\partial u}{\partial y} = x + 2z\frac{\partial z}{\partial y}$$

$$= x + 2zq$$

$$V = x + y + z$$

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x}$$

$$= 1 + p$$

$$\frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y}$$

$$= 1 + q$$

$$f(u,v) = 0$$

$$\begin{vmatrix} y + 2zp & 1 + p \\ x + 2zq & 1 + q \end{vmatrix} = 0 \quad \Rightarrow (y + 2zp)(1 + q) - (x + 2zq)(1 + p) = 0$$

$$\Rightarrow y + yq + 2zp + \cancel{2zpq} - x - xp - 2zq - \cancel{2zpq} = 0$$

$$\Rightarrow y - x + yq - xp + 2z(p - q) = 0.$$

Type 2:

2) $z = f(x+t) + g(x-t)$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'(x+t) + g'(x-t) \Rightarrow p = f'(x+t) + g'(x-t)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+t) + g''(x-t) \Rightarrow r = f''(x+t) + g''(x-t)$$

diff p.w.r.t 't'

$$\frac{\partial z}{\partial t} = f'(x+t) + g'(x-t)(-1) \Rightarrow q = f'(x+t) - g'(x-t)$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+t) - g''(x-t)(-1) \Rightarrow t = f''(x+t) + g''(x-t)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$$

Eliminating arbitrary constants.

4. $z = ax + by + \sqrt{a^2+b^2}$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

diff p.w.r.t 'y'

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b.$$

$$\therefore z = px + qy + \sqrt{p^2+q^2}$$

5. $z = (x^2+a^2)(y^2+b^2)$

diff p.w.r.t 'x'

$$\frac{\partial z}{\partial x} = 2x(y^2+b^2)$$

$$\Rightarrow \frac{p}{2x} = y^2+b^2$$

diff p.w.r.t 'y'

$$\frac{\partial z}{\partial y} = (x^2+a^2)(2y)$$

$$\Rightarrow \frac{q}{2y} = x^2+a^2$$

$$\therefore z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right) = \frac{pq}{4xy}.$$

$$\Rightarrow pq = 4xyz.$$

6. $(x-a)^2 + (y-b)^2 + z^2 = 1$

diff p.w.r.t 'x'

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

diff p.w.r.t 'y'

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$\Rightarrow 2(x-a) + 2zp = 0$

$2(x-a) = -2zp$

$x - a = -zp$

$2(y-b) + 2zq = 0$

$2(y-b) = -2zq$

$y - b = -zq$

$\therefore \quad (-zp)^2 + (-zq)^2 + z^2 = 1$

$\Rightarrow z^2p^2 + z^2q^2 + z^2 = 1$

$\Rightarrow (p^2 + q^2 + 1)z^2 = 1$