Linear PDE of II order with constant co-efficients

Homogeneous linear PDE:

A linear PDE with constant co-efficients in which all the partial derivatives are of the same order is called homogeneous; otherwise it is called non-homogeneous. Eg: i) $\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = \sin x$

ii) $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial z}{\partial x} + 7\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$

Homogeneous Linear Partial Differential Equations.

A homogeneous linear PDE is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{—(1)}$$

where $a_0, a_1, a_2, \dots, a_n$ are constants

(1) can be written as,

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n] z = F(x,y) \quad \text{—(2)}$$

where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

Solution of homogeneous linear PDE:

The complete solution of (2) is

$z =$ Complementary function + Particular Integral

$z = C.F + P.I$

To find C.F:

The C.F is the solution of equation

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n] z = 0$$

In the above equation, put $D \to m$ & $D' \to 1$,

then we get,

$$[a_0 m + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n] = 0.$$

which is called the auxillary equation.

Let the roots of this eq be $m_1, m_2, \dots, m_n$.

case i): If the roots are real (or imaginary) and different, then C.F is

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$$

case ii): If any two roots are equal say $m_1 = m_2 = m$ and others are different, then the C.F is

$$z = f_1(y+mx) + x f_2(y+mx) + f_3(y+m_3x) + \dots + f_n(y+m_nx)$$

case iii): If three roots are equal, say $m_1 = m_2 = m_3 = m$ then the C.F is

$$z = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx) + \dots + f_n(y+m_nx)$$

Hint: If in a homogeneous PDE, the RHS = 0 then $z = C.F$ [$P.I = 0$]

1. Solve: $(D^2 - 5DD' + 6D'^2)z = 0$ Put $D = m, D' = 1$

The A.E is $m^2 - 5m + 6 = 0$.

$\Rightarrow m_1 = 3, m_2 = 2$

$\Rightarrow$ The roots are real & different

$\therefore$ C.F $\Rightarrow z = f_1(y + m_1x) + f_2(y + m_2x)$

$z = f_1(y+3x) + f_2(y+2x)$

$P.I = 0$ [$\because RHS = 0$]

$\therefore z = C.F + P.I = f_1(y+3x) + f_2(y+2x)$

2. Solve $(D^2 - 6DD' + 9D'^2)z = 0$

Put $D \to m, D' \to 1$

$\therefore$ The A.E is $m^2 - 6m + 9 = 0$.

$\therefore m_1 = 3, m_2 = 3$.

$\Rightarrow$ The roots are real & equal.

C.F $\Rightarrow z = f_1(y+mx) + x f_2(y+mx)$

$z = f_1(y+3x) + x f_2(y+3x)$

$RHS = 0 \Rightarrow P.I = 0$.

$\therefore z = C.F + P.I = f_1(y+3x) + x f_2(y+3x)$

3. Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

$\Rightarrow (2D^2 + 5DD' + 2D'^2)z = 0$

put $D \to m$, $D' \to 1$

$\therefore 2m^2 + 5m + 2 = 0$

$\therefore m_1 = -2, \quad m_2 = -\frac{1}{2}$

$\Rightarrow$ The roots are real & different

C.F $\Rightarrow$ $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

RHS = 0 $\Rightarrow$ P.I = 0

$\therefore z = C.F + P.I = f_1(y-2x) + f_2(y - \frac{1}{2}x)$

To find Particular Integral (P.I):

Type - I: RHS $= f(x,y) = e^{ax+by}$

$$P.I = \frac{1}{\phi(D,D')} e^{ax+by}$$

Replace $D \to a$, $D' \to b$, then $P.I = \frac{1}{\phi(a,b)} e^{ax+by}$

provided $\phi(a,b) \neq 0$. If $\phi(a,b) = 0$, then diff the denominator w.r.t 'D' & multiply by x in numerator

1. Solve $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$

Given: $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

The A.E is $m^2 - 5m + 6 = 0$ [Replace $D \to m, D' \to 1$]

$m_1 = 2, m_2 = 3.$

$\Rightarrow$ The roots are real & different.

C.F is $z = f_1(y + 2x) + f_2(y + 3x)$

$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$ [Replace $D \to 1$, $D' \to 1$]

$= \frac{1}{1 - 5 + 6} e^{x+y} = \frac{1}{2} e^{x+y}$

$\therefore$ The solution is $z = C.F + P.I$

$$z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2}e^{x+y}$$

2. Solve: $(2D^2 - 2DD' + D'^2)z = 2e^{3y} + e^{x+y}$

The A.E is $2m^2 - 2m + 1 = 0.$

$$m^2 - m + \frac{1}{2} = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{2} = 0$$

$$\Rightarrow \left(m - \frac{1}{2}\right)^2 + \frac{1}{4} = 0 \quad \Rightarrow m - \frac{1}{2} = \pm i \frac{1}{\sqrt{4}} \Rightarrow m = \frac{1}{2} \pm \frac{i}{2}$$

$$\therefore m = \frac{1 \pm i}{2}$$

$\therefore$ The roots are imaginary and different

$\therefore$ C.F is $z = f_1\left(y + \left(\frac{1}{2} + \frac{i}{2}\right)x\right) + f_2\left(y + \left(\frac{1}{2} - \frac{i}{2}\right)x\right)$

$$P.I_1 = \frac{1}{2D^2 - 2DD' + D'^2} 2e^{3y}$$

Replace $D \to 0,\ D' \to 3.$

$$\therefore P.I_1 = \frac{1}{0 - 0 + 9} 2e^{3y} = \frac{2}{9}e^{3y}$$

$$P.I_2 = \frac{1}{2D^2 - 2DD' + D'^2} e^{x+y}$$

Replace $D \to 1,\ D' \to 1$

$$\therefore P.I_2 = \frac{1}{2 - 2 + 1} e^{x+y} = e^{x+y}$$

$\therefore z = C.F + P.I$

$$= f_1\left(y + \left(\frac{1}{2} + i\frac{1}{2}\right)x\right) + f_2\left(y + \left(\frac{1}{2} - i\frac{1}{2}\right)x\right) + \frac{2}{9}e^{3y} + e^{x+y}$$

3. Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Given $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

The A.E is $(m^2 - 4m + 4) = 0$.

$m_1 = 2, m_2 = 2$.

$\therefore$ The roots are real and equal

C.F is $z = f_1(y+2x) + x f_2(y+2x)$

P.I is $P.I_1 = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$

Replace $D \to 2$, $D' \to 1$

$$P.I_1 = \frac{1}{4-8+4} e^{2x+y}$$

$$= \frac{1}{0} e^{2x+y}.$$

diff dr w.r.t 'D' and multiply by x in Nr.

$$P.I_1 = \frac{x}{2D - 4D'} e^{2x+y}$$

$$= \frac{x}{2(2) - 4(1)} e^{2x+y} = \frac{x}{0} e^{2x+y}$$

diff dr w.r.t 'D' and multiply by x in Nr.

$$P.I = \frac{x^2}{2} e^{2x+y}.$$

$\therefore z = C.F + P.I = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$

4. Solve : $(D^2 - 3DD' + 2D'^2)z = e^{3x+2y}$ $[f_1(y+x) + f_2(y+2x) - e^{3x+2y}]$

5. Solve $(D^2 - DD' - 20D'^2)z = e^{5x+y}$ $[f_1(y-4x) + f_2(y+5x) + \frac{x}{9} e^{5x+y}]$

6. Solve $(D^2 + 2DD' + D'^2)z = e^{x-y}$ $[f_1(y-x) + x f_2(y-x) + \frac{x^2}{2} e^{x-y}]$

Type - II RHS = $\cos(ax+by)$ (or) $\sin(ax+by)$

Replace $D^2 \to -a^2$, $DD' \to -ab$, $D'^2 \to -b^2$

1. Solve $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

The A.E is $M^2 - 2M + 1 = 0$.

$M_1 = 1, M_2 = 1$

$\therefore$ The roots are real & equal.

$C.F = f_1(y+x) + x f_2(y+x)$

$P.I = \frac{1}{D^2-2DD'+D'^2} \cos(x-3y)$ $a = 1, b = -3$

Replace $D^2 \to -1^2$, $D'^2 \to -(-3)^2$, $DD' \to -(1)(-3)$

$D^2 \to -1$, $D'^2 \to -9$, $DD' \to 3$.

$= \frac{1}{-1-2(3)+(-9)} \cos(x-3y)$

$= \frac{1}{-16} \cos(x-2y)$

$\therefore z = C.F + P.I = f_1(y+x) + x f_2(y+x) - \frac{1}{16}\cos(x-3y)$.

2. Solve $(D^2 - 4D'^2)z = \sin(2x+y)$

The A.E is $m^2 - 4 = 0$.

$M_1 = 2, M_2 = -2$.

$\therefore$ The roots are real & different.

$C.F = f_1(y-2x) + f_2(y+2x)$.

$P.I = \frac{1}{D^2 - 4D'^2} \sin(2x+y)$ $a=2, b=1$

$= \frac{1}{-4-4(-1)} \sin(2x+y) = \frac{1}{0} \sin(2x+y)$

$= \frac{x}{2D} \sin(2x+y)$

$= \frac{x}{2}\left[\frac{-\cos(2x+y)}{2}\right]$

$$P.I = -\frac{x}{4}\cos(2x+y)$$

The soln is $\cancel{z} = C.F + P.I$

$$z = f_1(y-2x) + f_2(y+2x) - \frac{x}{4}\cos(2x+y)$$

3. Find the P.I of $(D^2 - 3DD' + D'^2)z = \sin x \cos y$

$$P.I = \frac{1}{D^2 - 3DD' + D'^2}\sin x \cos y. \quad [2\sin A \cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{2}\frac{1}{D^2 - 3DD' + D'^2}[\sin(x+y) + \sin(x-y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 3DD' + D'^2}\sin(x+y) + \frac{1}{D^2 - 3DD' + D'^2}\sin(x-y)\right]$$

$$P.I_1 = \frac{1}{D^2 - 3DD' + D'^2}\sin(x+y)$$

$a=1,\ b=1$

$D^2 \to -1,\ D'^2 \to -1,\ DD' \to -1$

$$= \frac{1}{-1-3(-1)+(-1)}\sin(x+y)$$

$$= \frac{1}{1}\sin(x+y) = \sin(x+y)$$

$$P.I_2 = \frac{1}{D^2 - 3DD' + D'^2}\sin(x-y)$$

$a=1,\ b=-1$

$D^2 \to -1,\ D'^2 \to -1,\ DD' \to 1$

$$= \frac{1}{-1-3(1)+(-1)}\sin(x-y)$$

$$= \frac{-1}{5}\sin(x-y)$$

$$\therefore P.I = \frac{1}{2}\left[\sin(x+y) - \frac{1}{5}\sin(x-y)\right]$$

$$= \frac{1}{2}\sin(x+y) - \frac{1}{10}\sin(x-y)$$

4. Find the P.I of $(D^2 + 4DD' - 5D'^2)z = \sin(x-2y)$

$$P.I = \frac{1}{D^2 + 4DD' - 5D'^2}\sin(x-2y)$$

$a=1,\ b=-2$

$D^2 \to -1,\ D'^2 \to -4$

$DD' \to 2$

$$= \frac{1}{-1+4(2)-5(-4)}\sin(x-2y)$$

$$= \frac{1}{27}\sin(x-2y)$$

5. Find the P.I of $(D^2+3DD'-4D'^2)z = \sin y$

6. Find the P.I of $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = \sin(3x+2y)$

Type - iii RHS $= x^m y^n$

1. Solve $(D^2-4DD'+4D'^2)z = xy$.

The A.E is $m^2-4m+4=0$

$(m-2)(m-2)=0$.

$m=2,2$.

$\therefore C.F = f_1(y+2x) + x f_2(y+2x)$

$$P.I = \frac{1}{D^2-4DD'+4D'^2} xy$$

$$= \frac{1}{D^2\left[1-\frac{4DD'}{D^2}+\frac{4D'^2}{D^2}\right]} xy$$

$$= \frac{1}{D^2\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]} xy \qquad (1-x)^{-1} = 1+x+x^2+\dots$$

$$= \frac{1}{D^2}\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)+\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)+\dots\right] xy$$

$$= \frac{1}{D^2}\left[xy+\frac{4D'}{D}(xy)-0\right]$$

$$= \frac{1}{D^2}\left[xy+\frac{4}{D}x\right]$$

$$= \frac{1}{D^2}xy + \frac{4}{D^3}x.$$

$$= \frac{x^3}{6}y + 4\frac{x^4}{24} = \frac{x^3 y}{6} + \frac{x^4}{6}$$

$\therefore z = C.F + P.I = f_1(y+2x) + x f_2(y+2x) + \frac{x^3 y}{6} + \frac{x^4}{6}$

2. Find the P.I of $(D^2 - DD' - 2D'^2)z = 2x + 3y$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2}(2x+3y)$$

$$= \frac{1}{D^2\left[1 - \frac{D'}{D} - \frac{2D'^2}{D^2}\right]}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)^2 + \dots\right](2x+3y)$$

$$= \frac{1}{D^2}\left[2x + 3y + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)(2x+3y)\right]$$

$$= \frac{1}{D^2}\left[2x + 3y + \frac{D'}{D}(2x+3y)\right]$$

$$= \frac{1}{D^2}\left[2x + 3y + \frac{1}{D}(3)\right]$$

$$= \frac{1}{D^2}[2x+3y] + \frac{1}{D^3}(3)$$

$$= \frac{x^3}{3} + \frac{3x^2 y}{2} + \frac{x^3}{2}$$

$$= \frac{3x^2 y}{2} + \frac{5x^3}{6}$$

RHS: $e^{ax+by} + \sin(ax+by)$ (or) $e^{ax+by} + \cos(ax+by)$

1. Solve $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x-y)$

The AE is $M^2 - M - 20 = 0$

$(M-5)(M+4) = 0 \Rightarrow M = 5, -4$

$C.F = f_1(y-4x) + f_2(y+5x)$.

$$P.I = \frac{1}{D^2 - DD' - 20D'^2}\left[e^{5x+y} + \sin(4x-y)\right]$$

$$= \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} + \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}, \quad a=5, b=1$$

$$= \frac{1}{5^2 - 5(1) - 20(1)^2} e^{5x+y} \quad = \frac{1}{0} e^{5x+y}$$

$$= \frac{x}{2D - D'} e^{5x+y}$$

$$= \frac{x}{2(5)-1} e^{5x+y} = \frac{x}{9} e^{5x+y}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y) \qquad a=4, b=-1$$

$$D^2 \to -16, D'^2 \to -1, DD' \to 4$$

$$= \frac{1}{-16 - 4 - 20(-1)} \sin(4x-y)$$

$$= \frac{x}{2D - D'} \sin(4x-y)$$

$$= x \frac{1}{2D - D'} \times \frac{2D + D'}{2D + D'} \sin(4x-y)$$

$$= \frac{x(2D+D') \sin(4x-y)}{4D^2 - D'^2}$$

$$= \frac{x(2D+D') \sin(4x-y)}{4(-16) - (-1)}$$

$$= -\frac{x}{63} [2D(\sin(4x-y)) + D' \sin(4x-y)]$$

$$= -\frac{x}{63} [2\cos(4x-y)(4) + \cos(4x-y)(-1)]$$

$$= -\frac{x}{63} [8\cos(4x-y) - \cos(4x-y)]$$

$$= -\frac{x}{63} 7\cos(4x-y)$$

$$= -\frac{x}{9} \cos(4x-y)$$

$\therefore\ z = C.F + P.I$

$$z = f_1(y-4x) + f_2(y+5x) + \frac{x}{9}e^{5x+y} - \frac{x}{9}\cos(4x-y)$$

2) $(D^2 + 4DD' - 5D'^2)z = e^{2x-y} + \sin(x-2y)$

3) $(D^2 - DD' - 30D'^2)z = xy + e^{6x+y}$

4) Solve $(D^2 - 2DD')z = e^{2x} + x^3y$.

The A.E is $m^2 - 2m = 0$

$m = 0, 2$

$\therefore$ C.F $= f_1(y) + f_2(y+2x)$.

$$P.I_1 = \frac{1}{D^2 - 2DD'} e^{2x} = \frac{1}{2^2 - 0} e^{2x} = \frac{1}{4} e^{2x}$$

$$P.I_2 = \frac{1}{D^2 - 2DD'} x^3y = \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1} x^3y$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2\right] x^3y$$

$$= \frac{1}{D^2}\left[x^3y + \frac{2}{D} x^3\right]$$

$$= \frac{1}{D^2}[x^3y] + \frac{2}{D^3}(x^3)$$

$$= \frac{x^5y}{20} + \frac{x^6}{60}$$

$$\therefore Z = C.F + P.I = f_1(y) + f_2(y+2x) + \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$$