Type - III $f(z,p,q)=0$.

Let $u = x+ay$

then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

1. Solve $p(1+q) = qz$

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Let $u = x+ay$, then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$1+a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az-1$$

$$\frac{dz}{du} = \frac{az-1}{a} \Rightarrow \frac{du}{dz} = \frac{a}{az-1}$$

$$du = \frac{a\,dz}{az-1}$$

Integrating $\int du = \int \frac{a\,dz}{az-1}$.

$$u = \log(az-1) + \log c$$

$$u = \log[c(az-1)]$$

2. Solve $z^2 = 1+p^2+q^2$

Let $u = x+ay$, then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

$$z^2 = 1+\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$z^2 = 1+\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z^2-1 = \left(\frac{dz}{du}\right)^2(1+a^2)$$

$$\frac{z^2-1}{1+a^2} = \left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du} = \frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$$

$$\frac{dz}{\sqrt{z^2-1}} = \frac{du}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\cosh^{-1}(z) = \frac{1}{\sqrt{1+a^2}} u + c$$

$$\cosh^{-1}(z) = \frac{x + ay}{\sqrt{1+a^2}} + c$$