X.

Method of Multipliers.

$$Pp + Qq = R \Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

I $$\frac{dx \pm dy \pm dz}{P \pm Q \pm R} = 0$$

II $$\frac{x\,dx \pm y\,dy \pm z\,dz}{xP \pm yQ \pm zR} = 0$$

III $$\frac{x^2\,dx \pm y^2\,dy \pm z^2\,dz}{x^2 P \pm y^2 Q \pm z^2 R} = 0$$

IV $$\frac{\frac{1}{x}\,dx \pm \frac{1}{y}\,dy \pm \frac{1}{z}\,dz}{\frac{1}{x} P \pm \frac{1}{y} Q \pm \frac{1}{z} R} = 0$$

v. $$\frac{\frac{1}{x^2}dx \pm \frac{1}{y^2}dy \pm \frac{1}{z^2}dz}{\frac{1}{x^2}P \pm \frac{1}{y^2}Q \pm \frac{1}{z^2}R} = 0$$

vi. $$\frac{l\,dx \pm m\,dy \pm n\,dz}{lP \pm mQ \pm nR} = 0$$

1. Solve $x(y-z)p + y(z-x)q = z(x-y)$

$Pp + Qq = R \Rightarrow P = x(y-z),\ Q = y(z-x),\ R = z(x-y)$

$\therefore$ The A.E is $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

I $$= \frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy}$$

$\Rightarrow dx+dy+dz = 0$

$\Rightarrow x+y+z = c_1$

ii $$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y}$$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$

$\Rightarrow \log x + \log y + \log z = \log c_2$

$\Rightarrow xyz = c_2$

$\therefore \phi(c_1, c_2) = 0$

$\Rightarrow \phi(x+y+z, xyz) = 0$

2. Solve $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

$Pp + Qq = R \Rightarrow P = x(z^2-y^2),\ Q = y(x^2-z^2),\ R = z(y^2-x^2)$

The A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

II $\dfrac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + z^2y^2 - x^2z^2}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$x^2 + y^2 + z^2 = c_1$

iv $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2}$.

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \log x + \log y + \log z = \log c_2$

$\Rightarrow xyz = c_2$

$\therefore \phi(c_1, c_2) = 0 \quad \Rightarrow \phi\,(x^2+y^2+z^2,\ xyz)$.

3. $(mz - ny)\,p + (nx - lz)\,q = ly - mx$

$Pp + Qq = R.$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

The A.E is $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}$

II $\dfrac{x\,dx + y\,dy + z\,dz}{xmz - xny + nxy - lzy + lzy - mxz}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2} \quad \Rightarrow c_1 = x^2 + y^2 + z^2$

vi $\dfrac{l\,dx + m\,dy + n\,dz}{lzm - lny + mnx - lmz + lny - mnx}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow lx + my + nz = c_2$

$\therefore \phi(c_1, c_2) = 0$

$\Rightarrow \phi(x^2+y^2+z^2,\ lx+my+nz) = 0$

4. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$Pp + Qq = R \quad \Rightarrow P = x^2(y-z),\ Q =$

The A.E is $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$

V $\dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y}$

$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz$

$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -c_1$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$

IV $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy-xz+yz-yx+zx-zy}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \log x + \log y + \log z = \log c_2$

$\Rightarrow xyz = c_2$

$\therefore \phi(c_1, c_2) = 0.$

$\therefore \phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ xyz\right) = 0$

5. Solve $z(x-y) = x^2p - y^2q$

$Pp + Qq = R \Rightarrow P = x^2,\quad Q = -y^2,\ R = z(x-y)$

The A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$$

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

$$\int \frac{dx}{x^2} = -\int \frac{dy}{y^2}$$

$$c_1 + \frac{-1}{x} = \frac{1}{y}.$$

$$c_1 = \frac{1}{x} + \frac{1}{y}$$

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{z(x-y)}$$

$$\frac{dx+dy}{(x+y)\cancel{(x-y)}} = \frac{dz}{z\cancel{(x-y)}}$$

$$\frac{dx+dy}{x+y} = \frac{dz}{z}$$

$$\frac{d(x+y)}{x+y} = \frac{dz}{z}$$

$$\log(x+y) = \log z + \log c_2$$

$$c_2 = \frac{x+y}{z}$$

$$\therefore \phi(c_1, c_2) = 0$$

$$\Rightarrow \phi\left(\frac{1}{x} + \frac{1}{y}, \frac{x+y}{z}\right) = 0$$

HW

6. Solve $(y^2+z^2)p - xyq + xz = 0$

$P = y^2+z^2, \quad Q = -xy, \quad R = -xz.$

The A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

$$= \frac{xdx + ydy + zdz}{xy^2 + xz^2 - xy^2 - xz^2}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$$\Rightarrow c_1 = x^2 + y^2 + z^2$$

Take $\frac{dy}{-xy} = \frac{dz}{-xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$

$$\Rightarrow \log y = \log z + \log c_2$$

$$\Rightarrow c_2 = \frac{y}{z}$$

$$\therefore \phi(c_1, c_2) = 0$$

$$\Rightarrow \phi\left(x^2+y^2+z^2, \frac{y}{z}\right) = 0$$

7. Solve $(x+2z)p+(2xz-y)q=x^2+y$.

$P=x+2z,\ Q=2xz-y,\ R=x^2+y$.

The A.E is $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$

$\Rightarrow \dfrac{dx}{x+2z}=\dfrac{dy}{2xz-y}=\dfrac{dz}{x^2+y}$ [Multipliers: $y, x, -2z$]

$\Rightarrow \dfrac{ydx+xdy-2zdz}{xy+2yz+2x^2z-xy-2zx^2-2zy}$

$\Rightarrow ydx+xdy-2zdz=0$

$\Rightarrow d(xy)-2zdz=0$

$\Rightarrow xy-z^2=c_1$

$\dfrac{dx}{x+2z}=\dfrac{dy}{2xz-y}=\dfrac{dz}{x^2+y}$ [Multipliers: $2x, -2, -2$]

$\Rightarrow \dfrac{2xdx-2dy-2dz}{2x^2+4xz-4xz+2y-2x^2-2y}$

$\Rightarrow 2xdx-2dy-2dz=0$

$\Rightarrow x^2-2y-2z=c_2$

$\therefore \phi(c_1,c_2)=0$

$\Rightarrow \phi(xy-z^2,\ x^2-2y-2z)=0$.