Parseval's Identity:

Let $F(s)$ be the Fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

Results: $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

1) S.T the Fourier tranform of $f(x) = \begin{cases} a^2-x^2, & |x|<a \\ 0, & |x|>a \end{cases}$
where $a>0$, and hence find that
$2\sqrt{\frac{2}{\pi}}\left[\dfrac{\sin as - as\cos as}{s^3}\right]$. Hence deduce that
$\displaystyle\int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$ using Parseval's identity and
show that $\displaystyle\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

$$f(x) = \begin{cases} a^2-x^2, & -a<x<a \\ 0, & -\infty<x<-a \ \ \& \ \ a<x<\infty. \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (a^2-x^2)(\cos sx + i\sin sx)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} \underbrace{(a^2-x^2)\cos sx}_{E \times E = E}\,dx + i\int_{-a}^{a} \underbrace{(a^2-x^2)\sin sx}_{E \times O = O}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_0^{a} (a^2-x^2)\cos sx\,dx + i(0)$$

$u = a^2-x^2$	$v = \cos sx$
$u' = -2x$	$v_1 = \dfrac{\sin sx}{s}$
$u'' = -2$	$v_2 = -\dfrac{\cos sx}{s^2}$
$u''' = 0$	$v_3 = -\dfrac{\sin sx}{s^3}$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]$$

Put $a=1$, $F_1(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

i) Using Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right][\cos sx - i\sin sx]\, ds$$

$$= \frac{2}{\pi}\, 2\int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\cos sx\, ds$$

Put $x=0$, $f(0) = \frac{4}{\pi}\int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$

$$\int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$$

ii) Using Parseval's identity:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-1}^{1} (1-x^2)^2\, dx = \int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds$$

$$2\int_0^1 (1+x^4-2x^2)\, dx = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[\frac{15+3-10}{15}\right] = \frac{16}{15}$$

$$\therefore \int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

Q. S.T the Fourier Transform of $f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$

& hence find that $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$. Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$.

Using Parseval's identity & show that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

$$f(x) = \begin{cases} 1-x^2, & -1<x<1 \\ 0, & -\infty<x<-1 \ \& \ 1<x<\infty. \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)(\cos sx + i\sin sx)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} \underbrace{(1-x^2)}_{E}\times\underbrace{\cos sx}_{E} = E\, dx + i\int_{-1}^{1} \underbrace{(1-x^2)}_{E} \times \underbrace{\sin sx}_{O} = O\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_0^1 (1-x^2)\cos sx\, dx$$

$u = 1-x^2$	$V = \cos sx$
$u' = -2x$	$V_1 = \frac{\sin sx}{s}$
$u'' = -2$	$V_2 = -\frac{\cos sx}{s^2}$
$u''' = 0$	$V_3 = -\frac{\sin sx}{s^3}$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[(1-x^2)\frac{\sin sx}{s} - 2x\frac{\cos sx}{s^2} + 2\frac{\sin sx}{s^3}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[-2\frac{\cos s}{s^2} + \frac{2\sin s}{s^3}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

i) Using Inverse Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

$$1-x^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right][\cos sx - i\sin sx]\, ds$$

$$1-x^2 = \frac{2}{\pi} 2\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos sx \, ds$$

Put $x=0$.

$$1 = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3} \, ds$$

$$\therefore \int_0^\infty \frac{\sin s - s\cos s}{s^3} \, ds = \frac{\pi}{4}$$

ii) Using Parseval's identity:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \frac{\sin s - s\cos s}{s^3}\right]^2 ds$$

$$2\int_0^1 (1+x^4-2x^2) dx = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15+3-10}{15}\right]$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds. = \frac{16}{15}$$

$$\therefore \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

3. Find the F.T of $f(x) = \begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a>0 \end{cases}$ and deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ & $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$

$$f(x) = \begin{cases} a-|x|, & -a<x<a \\ 0, & -\infty<x<-a \ \& \ a<x<\infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$u = a-x \quad V = \cos sx$

$u' = -1 \quad V_1 = \frac{\sin sx}{s}$

$u'' = 0 \quad V_2 = -\frac{\cos sx}{s^2}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\,[\cos sx + i \sin sx]\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sa}{s^2} + \frac{1}{s^2} \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos sa}{s^2} \right] = \sqrt{\frac{2}{\pi}}\ \frac{2 \sin^2 \frac{sa}{2}}{s^2} \quad \left[\begin{array}{l} \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta) \end{array} \right]$$

i) Using Inverse Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$a-|x| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{s^2} 2 \sqrt{\frac{2}{\pi}} \sin^2 \frac{sa}{2}\, e^{-isx}\, ds$$

$$a-|x| = \frac{2}{\pi}\, 2 \int_0^\infty \frac{1}{s^2} \sin^2 \frac{sa}{2}\, [\cos sx - i \sin sx]\, ds \quad \cancel{\cos sx\, ds}$$

$$a-|x| = \frac{4}{\pi} \int_0^\infty \frac{1}{s^2} \sin^2 \frac{sa}{2} \cos sx\, ds.$$

Put $x = 0,\ a = 2$

$$2 = \frac{4}{\pi} \int_0^\infty \frac{1}{s^2} \sin^2 s \, ds.$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{2\pi}{4} \quad \Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

iii) Parseval's identity :

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

$$\int_{-2}^{2} [2-|x|]^2 dx = \int_{-\infty}^{\infty} 4 \frac{2}{\pi} \frac{\sin^4 \frac{as}{2}}{s^4} ds.$$

$$2\int_0^2 (2-x)^2 dx = \frac{8}{\pi} 2\int_0^{\infty} \left(\frac{\sin \frac{sa}{2}}{s}\right)^4 ds.$$

$$2\int_0^2 4 + x^2 - 4x \, dx = \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^4 ds$$

$$2\left[4x + \frac{x^3}{3} - \frac{4x^2}{2}\right]_0^2 = \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^4 ds$$

$$2\left[8 + \frac{8}{3} - 8\right] = \frac{16}{\pi} \int_0^{\infty} \left[\frac{\sin \frac{as}{2}}{s}\right]^4 ds$$

$$\frac{16}{3} \times \frac{\pi}{16} = \int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^4 ds$$

$$\int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^4 ds = \frac{\pi}{3}$$

$$\because a = 2 \quad \Rightarrow \int_0^{\infty} \left(\frac{\sin s}{s}\right)^4 ds = \frac{\pi}{3}$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

4. Find the F.T of $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ and deduce that

$\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$ & $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt$.

$$f(x) = \begin{cases} 1-|x|, & -1 < x < 1 \\ 0, & -\infty < x < -1 \;\&\; 1 < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i\sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} 2\int_0^1 (1-x)(\cos sx) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x) \cos sx \, dx$$

$u = 1-x$ $\quad V = \cos sx$

$u' = -1$ $\quad V_1 = \frac{\sin sx}{s}$

$u'' = 0$ $\quad V_2 = \frac{-\cos sx}{s^2}$

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{s}{2}}{s^2}\right]$$

i) Using Inverse Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$1-|x| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\frac{\sin^2 s/2}{s^2}(\cos sx - i\sin sx)\,ds$$

$$1-|x| = \frac{2}{\pi}2\int_0^{\infty}\frac{1}{s^2}\sin^2\frac{s}{2}\cos sx\,ds.$$

$$1-|x| = \frac{4}{\pi}\int_0^{\infty}\frac{1}{s^2}\sin^2\frac{s}{2}\cos sx\,ds$$

Put $x = 0,\ s = 2t \Rightarrow ds = 2dt$

$$\Rightarrow 1 = \frac{4}{\pi}\int_0^{\infty}\frac{1}{(2t)^2}\sin^2 t\,2dt$$

$$\Rightarrow 1 = \frac{2}{\pi}\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt$$

$$\Rightarrow \int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2} \quad \Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

ii) Using Parseval's identity:

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|F(s)|^2\,ds.$$

$$\int_{-1}^{1}(1-|x|)^2\,dx = \int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\frac{\sin^2\frac{s}{2}}{s^2}\right]^2 ds$$

$$2\int_0^1(1-x)^2\,dx = \frac{8}{\pi}2\int_0^{\infty}\frac{\sin^4 t}{(2t)^4}\,2dt$$

$$\int_0^1(1+x^2-2x)\,dx = \frac{16}{16\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$$

$$\left[x + \frac{x^3}{3} - \frac{2x^2}{2}\right]_0^1 = \frac{1}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$$

$$1+\frac{1}{3}-1 = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$$

$$\frac{\pi}{3} = \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$