Fourier Sine and Cosine transform.

* Fourier Sine transform:

Fourier sine transform of $f(x)$ is defined as

$$F_S(s) = F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx.$$

The inverse Fourier sine transform of $f(x)$ is defined as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds.$$

* Fourier Cosine transform:

Fourier cosine transform of $f(x)$ is defined as

$$F_C(s) = F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

The inverse Fourier cosine transform of $f(x)$ is defined as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds.$$

~~Properties~~

1) Find the F.S.T of $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$

$$F_S(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s} + \frac{1}{s} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos s}{s} \right]$$

2. Find the F.S.T of $f(x) = e^{-ax}$, $a > 0$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2} \right] \qquad \because \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

3. Find the F.C.T of $5e^{-2x} + 2e^{-5x}$

$$F_c[5e^{-2x}+2e^{-5x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty (5e^{-2x}+2e^{-5x}) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^\infty 5e^{-2x} \cos sx \, dx + \int_0^\infty 2e^{-5x} \cos sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[5 \int_0^\infty e^{-2x} \cos sx \, dx + 2\int_0^\infty e^{-5x} \cos sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[5 \left(\frac{2}{s^2+2^2}\right) + 2\left(\frac{5}{s^2+5^2}\right)\right]$$

$$= 10\sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2+4} + \frac{1}{s^2+25}\right]$$

4. Find F.C.T of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[\cos x] = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos sx \cos x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{\cos(s+1)x + \cos(s-1)x}{2} dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^a \cos(s+1)x \, dx + \sqrt{\frac{2}{\pi}} \int_0^a \cos(s-1)x \, dx\right]$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right]$$