# Properties of Fourier Transform, FST & FCT.

## 1) Linear property:

i) $F[a f(x) + b g(x)] = a F[f(x)] + b F[g(x)]$

ii) $F_s[a f(x) + b g(x)] = a F_s[f(x)] + b F_s[g(x)] = a F_s(s) + b G_s(s)$

iii) $F_c[a f(x) + b g(x)] = a F_c[f(x)] + b F_c[g(x)] = a F_c(s) + b G_c(s)$

Proof:

ii) $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$F_s[a f(x) + b g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty [a f(x) + b g(x)] \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty a f(x) \sin sx \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty b g(x) \sin sx \, dx$$

$$= a F_s[f(x)] + b F_s[g(x)] = a F_s(s) + b G_s(s)$$

iii) $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$$F_c[a f(x) + b g(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty [a f(x) + b g(x)] \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty a f(x) \cos sx \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty b g(x) \cos sx \, dx$$

$$= a F_c[f(x)] + b F_c[g(x)] = a F_c(s) + b G_c(s)$$

2) Change of scale property:

i) $F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$

ii) $F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$

Proof:

i) $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \cos s x \, dx.$

$ax = y$

$x = y/a \Rightarrow dx = dy/a$

$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos s(y/a) \frac{dy}{a}$

$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos\left(\frac{s}{a}\right) y \, dy$

$= \frac{1}{a} F_c\left[\frac{s}{a}\right]$

ii) $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \sin sx \, dx$

$\Rightarrow ax = y, \quad x = \frac{y}{a} \Rightarrow dx = \frac{dy}{a}$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin s\left(\frac{y}{a}\right) \frac{dy}{a}$

$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin\left(\frac{s}{a}\right) y \, dy$

$= \frac{1}{a} F_s\left(\frac{s}{a}\right)$

3. Shifting Property:

i) $F[f(x-a)] = e^{ias} F(s)$

ii) $F[e^{iax} f(x)] = F[s+a]$

4. Modulation property:

If $F(s)$ is the Fourier transform of $f(x)$ then

i) $F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

Proof: ii) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c[s+a] + F_c(s-a)]$

iii) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

iv) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

v) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a) + F_s(a-s)]$

Proof:

i) $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{iax} + e^{-iax}}{2} e^{isx} dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx + \int_{-\infty}^{\infty} f(x) e^{i(s-a)x} dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{+i(s-a)x} dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$

ii) $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \cos ax \, dx$$

$[\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]]$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \frac{1}{2}[\cos(s+a)x + \cos(s-a)x] dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(s+a)x \, dx + \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(s-a)x \, dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

5. i) $F_s[f'(x)] = -s F_c(s)$, if $f(x) \to 0$ as $x \to \infty$

ii) $F_c[f'(x)] = \sqrt{\frac{2}{\pi}} f(0) + s F_s(s)$, if $f(x) \to 0$ as $x \to \infty$

6. Identity property:

i) $\int_0^\infty f(x) g(x)\, dx = \int_0^\infty F_c(s) G_c(s)\, ds$

ii) $\int_0^\infty f(x) g(x)\, dx = \int_0^\infty F_s(s) G_s(s)\, ds$

iii) $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_s(s)|^2\, ds$

iv) $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, ds.$

7. i) $F_s[x f(x)] = -\frac{d}{ds} F_c(s)$

ii) $F_c[x f(x)] = -\frac{d}{ds} F_s(s)$

Proof: $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$

Diff. on B.S w.r.t 's'

$$\frac{d}{ds} F_c[f(x)] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s} (\cos sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) (-\sin sx\,(x))\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \sin sx\, dx$$

$$= -F_s[x f(x)]$$

$$\Rightarrow F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)] = -\frac{d}{ds} F_c(s).$$

$$\int_0^\infty e^{-ax} \cos sx \, dx = \frac{a}{s^2+a^2} \quad \& \quad \int_0^\infty e^{-ax} \sin sx \, dx = \frac{s}{s^2+a^2}$$