Linear PDE of 2^{nd} ~~and higher~~ order with constant coefficients.

Homogeneous Linear PDEs.:

A linear PDE with constant coefficients in which all the partial derivatives are of the same order is called homogeneous; otherwise it is called non-homogeneous.

Example:

Homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = \sin x$$

Non-homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial z}{\partial x} + 7\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$$

Notation:

$$D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$$

Method of finding complementary function (CF)

Let the given equation be of the form

$$f(D, D')z = f(x, y).$$

Put $D = m$
$D' = 1$

$f(m,1) = 0 \Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$

Let the roots of the eqn. be $m_1, m_2, \dots, m_n$

Roots	complementary function
1. The roots are different. $m_1, m_2, \dots, m_n$	$CF = f_1(y+m_1x) + f_2(y+m_2x) + \dots + f_n(y+m_nx)$
2. The roots are equal. $m_1 = m_2 = \dots = m_n = m$ (say)	$CF = f_1(y+mx) + x f_2(y+mx) + \dots + x^{n-1} f_n(y+mx)$

General Solution is $y = CF + PI$

RHS $= 0$ $\quad (z = CF)$

1. Solve $(D^2 - 6DD' + 9D'^2)z = 0$

Soln.:

Put $D = m, \; D' = 1$

The auxillary equation is,

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3, 3$ (equal roots)

The solution is

$z = CF$

$= f_1(y+3x) + x f_2(y+3x)$

RHS $= e^{ax+by}$

Replace D by a

D' by b

1. Solve $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Soln:

The auxiliary equation is

$m^2 - 5m + 6 = 0$ $\quad (D \to m$, $D' \to 1)$

$(m-3)(m-2) = 0$

$m = 2, 3$ (Different)

$$\begin{array}{c} 6 \\ -3 \quad -2 \end{array}$$

$CF = f_1(y+2x) + f_2(y+3x)$

$$PI = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$$

Replace

$D \to a = 1$

$D' \to b = 1$

$$= \frac{1}{1-5+6} e^{x+y}$$

$$= \frac{1}{2} e^{x+y}$$

The Solution is, $z = CF + PI$

$= f_1(y+2x) + f_2(y+3x) + \frac{e^{x+y}}{2}$

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2. Solve $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

Soln:

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$ (Equal)

$$\begin{array}{c} 4 \\ -2 \quad -2 \end{array}$$

$CF = f_1(y+2x) + x f_2(y+2x)$

$$PI = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

$$= \frac{1}{2^2 - 4(2)(1) + 4(1)^2} e^{2x+y}$$

Replace
$D \to a = 2$
$D' \to b = 1$

$$= \frac{1}{4-8+4} e^{2x+y}$$

[multiply x in the Nr & Differentiate Dr w.r.t D]

$$= x \frac{1}{2D - 4D'} e^{2x+y}$$

$$= x^2 \frac{1}{2} e^{2x+y}$$

$D \to 2$
$D' \to 1$

$$= \frac{x^2}{2} e^{2x+y}$$

The Solution is $z = CF + PI$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

HW

Solve

i]. $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x-y}$

2]. ~~Solve~~ Find the PI of

$(D^2 + DD')z = e^{x-y} + e^{x+y}$

RHS $= \cos(ax+by)$ or $\sin(ax+by)$

Replace $D^2 \to -a^2$

$DD' \to -ab$

$D'^2 \to -b^2$

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1. Solve $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

Soln. :

AE $m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$ (equal)

$CF = f_1(y+x) + x f_2(y+x)$

$PI = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y)$

$a = 1, b = -3$

$D^2 \to -a^2 = -1^2 = -1$

$DD' \to -ab = -1(-3) = 3$

$D'^2 \to -b^2 = -(-3)^2 = -9$

$= \frac{1}{-1-2(3)-9} \cos(x-3y)$

$= \frac{-1}{16} \cos(x-3y)$

$\therefore z = CF + PI$

$= f_1(y+x) + x f_2(y+x) - \frac{1}{16}\cos(x-3y)$

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2. Solve $(D^2 - 4D'^2)z = \sin(2x+y)$

Soln. :

AE $m^2 - 4 = 0$

$(m+2)(m-2) = 0$

$m = -2, 2$ (different)

$CF = f_1(y-2x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 - 4D'^2} \sin(2x+y)$$

$a = 2$
$b = 1$

$D^2 \to -a^2 = -2^2$
$= -4$

$$= \frac{1}{-4-4(-1)} \sin(2x+y)$$

$DD' \to -ab = -2$
$= -2$

$$= x \frac{1}{2D} \sin(2x+y)$$

$D'^2 \to -b^2 = -1^2$
$= -1$

$$= \frac{x}{2}\left(\frac{-\cos(2x+y)}{2}\right)$$

$$PI = -\frac{x}{4} \cos(2x+y)$$

$\therefore$ The Soln. is $z = CF + PI$

$$z = f_1(y-2x) + f_2(y+2x) - \frac{x}{4}\cos(2x+y)$$

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3. Find the PI of $(D^2 - 3DD' + D'^2)z = \sin x \cos y$

Soln. :

$$PI = \frac{1}{D^2 - 3DD' + D'^2} \sin x \cos y$$

Gvn. $(D^2 - 3DD' + D'^2)z = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$$PI = \frac{1}{2}\left[\frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) + \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y)\right]$$

$$= \frac{1}{2}[PI_1 + PI_2] \to (1)$$

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$$PI_1 = \frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) \qquad a=1,\ b=1$$

$$= \frac{1}{-1-3(-1)-1} \sin(x+y) \qquad D^2 \to -a^2 = -1$$

$$DD' \to -ab = -1$$

$$= \frac{1}{-2+3} \sin(x+y) \qquad D'^2 \to -b^2 = -1$$

$$= \sin(x+y)$$

$$PI_2 = \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y) \qquad a=1,\ b=-1$$

$$D^2 \to -a^2 = -1$$

$$= \frac{1}{-1-3(1)-1} \sin(x-y) \qquad DD' \to -ab = -1(-1) = 1$$

$$D'^2 \to -b^2 = -(-1)^2 = -1$$

$$= \frac{1}{-5} \sin(x-y)$$

$$(1) \Rightarrow \quad PI = \frac{1}{2}\left[\sin(x+y) - \frac{1}{5}\sin(x-y)\right]$$

$$= \frac{1}{2}\sin(x+y) - \frac{1}{10}\sin(x-y)$$

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4) ✓ find the PI of $(D^2 + 4DD' - 5D'^2)z = \sin(x-2y)$

Soln.:

$$PI = \frac{1}{D^2 + 4DD' - 5D'^2} \sin(x-2y) \qquad a=1,\ b=-2$$

$$= \frac{1}{1+4(2)-5(-4)} \sin(x-2y) \qquad D^2 \to -a^2 = -1 = 1$$

$$DD' \to -ab = -1(-2) = 2$$

$$= \frac{1}{29} \sin(x-2y) \qquad D'^2 \to -b^2 = -(-2)^2 = -4$$

HW

1. Find the PI of $(D^2+3DD'-4D'^2)z = \sin y$

2. Find the PI of $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2}$

$= \sin(3x+2y)$

RHS $= x^m y^n$

1. Solve $(D^2-4DD'+4D'^2)z = xy$

Soln.

AE $\quad m^2-4m+4=0$

$(m-2)(m-2)=0$

$m = 2, 2$ (Equal)

$CF = f_1(y+2x) + xf_2(y+2x)$

$$PI = \frac{1}{D^2-4DD'+4D'^2}xy$$

$$= \frac{1}{D^2\left[1-\frac{4DD'}{D^2}+\frac{4D'^2}{D^2}\right]}xy$$

$$= \frac{1}{D^2\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]}xy$$

$$= \frac{1}{D^2}\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]^{-1}xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)+\cdots\right]xy$$

$(\because (1-x)^{-1} = 1+x+x^2+\cdots$

$$= \frac{1}{D^2}\left[xy + \frac{4D'}{D}(xy) - 0\right]$$

$$= \frac{1}{D^2}\left[xy + \frac{4}{D}x\right]$$

$$= \frac{1}{D^2}xy + \frac{4}{D^3}x$$

$$\frac{1}{D^2}xy \xrightarrow{1st} \frac{1}{D}\frac{x^2}{2}y \to \frac{x^3}{6}y$$

$$\frac{1}{D^3}x \to \frac{1}{D^2}\frac{x^2}{2} \to \frac{1}{D}\frac{x^3}{6} \to \frac{x^4}{24}$$

$$= \frac{x^3y}{6} + 4\frac{x^4}{24}$$

$$= \frac{x^3y}{6} + \frac{x^4}{6}$$

$\therefore$ The Solution is, $z = CF + PI$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^3y}{6} + \frac{x^4}{6}$$

2] Find the PI of $(D^2 - DD' - 2D'^2)z = 2x+3y$

Soln. :

$$PI = \frac{1}{D^2 - DD' - 2D'^2}(2x+3y)$$

$$= \frac{1}{D^2\left[1 - \frac{D'}{D} - \frac{2D'^2}{D^2}\right]}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right) + \cdots\right](2x+3y)$$

$$= \frac{1}{D^2}\left[2x+3y + \frac{D'}{D}(2x+3y)\right]$$

$$= \frac{1}{D^2}\left[(2x+3y) + \frac{1}{D}(3)\right]$$

$$= \frac{1}{D^2}\left[(2x+3y) + \frac{3}{D}\right]$$

$$= \frac{1}{D^2}(2x+3y) + \frac{3}{D^3}$$

$$\frac{1}{D^2}(2x+3y) = \frac{1}{D}\left[2\frac{x^2}{2} + 3xy\right]$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2}$$

$$\frac{1}{D^3} = \frac{1}{D^2}x = \frac{1}{D}\frac{x^2}{2} = \frac{x^3}{6}$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + 3\frac{x^3}{6}$$

$$PI = \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{x^3}{2}$$

RHS $= e^{ax+by} + \sin(ax+by)$,

(or)

$e^{ax+by} + \cos(ax+by)$

1. Solve $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x-y)$

Soln.

AE $\quad m^2 - m - 20 = 0 \qquad D \to m$

$(m+5)(m+4) = 0 \qquad D' \to 1$

$m = 5, -4$

$CF = f_1(y-4x) + f_2(y+5x)$

$$PI = \frac{1}{D^2 - DD' - 20D'^2}\left[e^{5x+y} + \sin(4x-y)\right]$$

$$= \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} + \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$PI = PI_1 + PI_2$

$$PI_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$$

$$= \frac{1}{25 - 5(1) - 20(1)^2} e^{5x+y} \qquad D \to a = 5, \quad D' \to b = 1$$

$$= \frac{1}{0} e^{5x+y}$$

$$= x \frac{1}{2D - D'} e^{5x+y}$$

$$= x \frac{1}{2(5) - 1} e^{5x+y}$$

$$PI_1 = \frac{x}{9} e^{5x+y}$$

$a = 4, \quad b = -1$

$$PI_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$D^2 \to -a^2 = -16$

$DD' \to -ab = -4(-1) = 4$

$D'^2 \to -b^2 = -(-1)^2 = -1$

$$= \frac{1}{-16 - 4 - 20(-1)} \sin(4x-y)$$

$$= x \frac{1}{2D - D'} \sin(4x-y)$$

$$= x \frac{(2D + D')}{(2D - D')(2D + D')} \sin(4x-y)$$

$$= x \frac{(2D + D') \sin(4x-y)}{4D^2 - D'^2}$$

$$= \frac{x}{-64 + 1} (2D + D') \sin(4x-y)$$

$$= -\frac{x}{63} \left[2D \sin(4x-y) + D' \sin(4x-y) \right]$$

$$= -\frac{x}{63}\left[8\cos(4x-y) - \cos(4x-y)\right]$$

$$= \frac{-7x}{63}\cos(4x-y)$$

$$= -\frac{x}{9}\cos(4x-y)$$

The soln. is, $z = CF + PI$

$$z = f_1(y-4x) + f_2(y+5x) + \frac{x}{9}e^{5x+y} - \frac{x}{9}\cos(4x-y)$$

HW

1. $(D^2+4DD'-5D'^2)z = e^{2x-y} + \sin(x-2y)$
2. $(D^2-DD'-30D'^2)z = xy + e^{6x+y}$

Solve $r+s-6t = y\cos x$

Soln. :

Given $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

$$(D^2+DD'-6D'^2)z = y\cos x$$

AE $\quad m^2+m-6=0$

$$(m+3)(m-2)=0$$

-6: 3, -2

$$m = -3, 2$$

$$CF = f_1(y-3x) + f_2(y+2x)$$

$$PI = \frac{1}{(D^2+DD'-6D'^2)}\, y\cos x$$

factor $\rightarrow D-2D'$

where $y = c - 2x$

$D \rightarrow c$

$D' \rightarrow x$

$$= \frac{1}{(D+3D')(D-2D')} y \cos x$$

$$= \frac{1}{(D+3D')} \int (c-2x) \cos x \, dx$$

$$= \frac{1}{D+3D'} \left[(c-2x) \sin x - (-2)(-\cos x) \right]$$

$$= \frac{1}{D+3D'} \left[y \sin x - 2\cos x \right]$$ factor $\rightarrow D+3D'$

$y \rightarrow c+3x$

$$= \int \left[(c+3x) \sin x - 2\cos x \right] dx$$

$$= (c+3x)(-\cos x) - 3(-\sin x) - 2\sin x$$

$$= -y\cos x + 3\sin x - 2\sin x$$

$$= -y\cos x + \sin x$$