Theorem : 6

A connected graph is Eulerian iff evy. vertex is of even degree.

Proof:

Let G be an Eulerian graph.
we've to prove, all the vertices are of even degree
Since G is Eulerian, G contains an Euler circuit
$v_0\, e_1\, v_1\, e_2\, v_2, \dots\, e_{n-1}\, v_{n-1}\, e_n\, v_0$.

Both the edges e_1 and e_n that contributes 1 to the degree of v_0 and so degree of v_0 is atleast 2.

In tracing this circuit, we can find an edge enters the vertex and another edge leaves the vertex, contribute 2 to the degree of vertex.

This is true for all the vertices and so each vertex is of degree 2, which is an even no.

If G is an Eulerian graph, then each vertex of the graph is of even degree.

Conversely, let G be a graph such that all the vertices are of even degree.

To prove: G is an Euler graph.

we shall construct an Euler circuit and to prove G is an Euler graph.

Let v be the arbitrary vertex in G.

Beginning with v form a circuit $C: v, v_1, v_2, \dots v_{n-1}, v$

This is possible becoz each vertex is of even degree, we can leave a vertex along an edge not used to enter it.

This tracing stops only at the vertex becoz V is also of even degree and we started from V. Thus we get a circuit.

If C includes all the edges of C, then C is an Euler circuit.

Hence G is Eulerian.

Suppose C does not contain all the edges of G. consider the subgraph H of G by deleting all the edges of C from G and vertices not incident with the remaining edges.

Note that all the vertices of H are of even degree. Since G is connected, H and C must have a common vertex u.

Beginning with u, construct a circuit C_1 for H. Now combine C and C_1 to form a larger circuit C_2. If it contains all the edges of G, then G is Eulerian.

otherwise continue this process until we get an Eulerian circuit.

Since G is finite, this procedure will come to the end with an Eulerian circuit.

Hence G is Eulerian.