Properties of Fourier Transform, FST and FCT:

I. Linear Property:

→ FT $F[af(x)+bg(x)] = aF[f(x)] + bF[g(x)]$ where a and b are real numbers.

proof:

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]\,e^{isx}\,dx$$

$$= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{isx}\,dx$$

$$= a\,F[f(x)] + b\,F[g(x)]$$

→ FST $F_S[af(x)+bg(x)] = a F_S[f(x)] + b F_S[g(x)]$

proof:

$$F_S[af(x)+bg(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}[af(x)+bg(x)]\sin sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\,dx + b\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(x)\sin sx\,dx$$

$$= a\,F_S[f(x)] + b\,F_S[g(x)]$$

III→ FCT $F_C[af(x)+bg(x)] = a F_C[f(x)] + b F_C[g(x)]$

2]. change of scale property:

For any non-zero real a, $F[f(ax)] = \frac{1}{|a|} F\left[\frac{s}{a}\right]$, $a>0$

proof:

WKT, $$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

Now, $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$

put $t = ax$ | when $x = -\infty \Rightarrow t = -\infty$

$\frac{dt}{dx} = a$ | $x = \infty \Rightarrow t = \infty$

$\Rightarrow dx = \frac{dt}{a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}}\, \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{+is(t/a)}\, dt$$

$i\left(\frac{s}{a}\right)t$

$$= \frac{1}{a} F\left[\frac{s}{a}\right]$$

3]. Shifting Property:

i). $F[f(x-a)] = e^{ias} F(s)$

ii). $F[e^{iax} f(x)] = F(s+a)$

proof:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

i). Now, $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$

put $t = x - a$ | $x = -\infty \Rightarrow t = -\infty$

$dt = dx$ | $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = e^{ias} F(s)$$

ii). $F[e^{iax} f(x)] = F[s+a]$

Proof:

Now $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F(s+a)$$

4). Modulation Property:

ST: If $F(s)$ is the Fourier transform of $f(x)$

then $F[f(x) \cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

proof:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Now, $F[f(x) \cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\frac{e^{iax} + e^{-iax}}{2}\right] e^{isx}\, dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{i(s+a)x} + e^{i(s-a)x}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

FST →

$$\cancel{F_s[f(x) \sin ax] = \frac{1}{2}}$$

$$F_s[f(x)\cos ax] = \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$$

Proof:

WKT $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x) \sin sx\, dx$

Now, $F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x) \cos ax \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x) \sin \underbrace{sx}_{A} \cos \underbrace{ax}_{B}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x) \frac{1}{2}\left[\sin \underbrace{(sx+ax)}_{(A+B)} + \sin(sx-ax)\right] dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x) \sin(s+a)x\, dx + \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x) \sin(s-a)x\, dx\right]$$

$$= \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$$