Lagrange's Linear Equation:

The equation is of the form

$Pp + Qq = R$, where P, Q and R are functions of x, y, z. This is known as Lagrange's linear eqn.

To solve this equation, it is enough to solve the subsidiary (or) auxillary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The auxillary eqn. can be solved in two ways.

i). Method of grouping

ii) Method of multipliers.

Method of grouping:

In the auxillary eqn., $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

If the variables can be separated in any pair of eqns., then we get a solution is of the form

$u(x, y) = c_1$ and $v(x, y) = c_2$

i.e., $\phi(u, v) = 0$ where ϕ is arbitrary.

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1. Solve $px^2 + qy^2 = z^2$

Soln.:

$$px^2 + qy^2 = z^2$$

This eqn. is of the form

$Pp + Qq = R$ where $P = x^2$, $Q = y^2$

and $R = z^2$

The auxillary eqn. is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Take $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating, we get

$$\int x^{-2}\,dx = \int y^{-2}\,dy$$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

$$-\frac{1}{x} + \frac{1}{y} = c_1$$

$$u = \frac{1}{y} - \frac{1}{x}$$

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

$$\int y^{-2}\,dy = \int z^{-2}\,dz$$

$$\frac{y^{-1}}{-1} = \frac{z^{-1}}{-1} + c_2$$

$$\frac{-1}{y} + \frac{1}{z} = c_2$$

$$V = \frac{1}{z} - \frac{1}{y}$$

$\therefore$ The soln. is $\phi(u, v) = 0$

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$$

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3. Solve $\frac{y^2 z}{x} p + xzq = y^2$

Soln. :

$$\frac{y^2 z}{x} p + xzq = y^2 \qquad P = \frac{y^2 z}{x}, \quad Q = xz$$

$$R = y^2$$

AE $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$$x^2\,dx = y^2\,dy$$

Integrating,

$$\frac{x^3}{3} = \frac{y^3}{3} + c_3$$

$$\frac{x^3}{3} - \frac{y^3}{3} = c_3$$

$$x^3 - y^3 = 3c_3 = u$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$$x\,dx = z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$x^2 - z^2 = 2c_2$$

$$v = x^2 - z^2$$

$\therefore$ The soln. is $\phi(u, v) = 0$

$$\phi(x^3 - y^3,\ x^2 - z^2) = 0$$

3]. Solve $px + qy = x$

Soln.

$$Pp + Qq = R \Rightarrow P = x,\quad Q = y,\quad R = x$$

AE $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \Rightarrow u = \frac{x}{y}$$

$$\frac{dx}{x} = \frac{dz}{x}$$

$$dx = dz$$

$$\int dx = \int dz$$

$$x = z + c_2$$

$$x - z = c_2$$

$$\Rightarrow v = x - z$$

$$\Rightarrow \phi\left(\frac{x}{y},\ x - z\right) = 0$$

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4]. Solve $p\tan x + q\tan y = \tan z$

Soln.:

$$Pp + Qq = R \Rightarrow P = \tan x,\quad Q = \tan y,\quad R = \tan z$$

AE
$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating,

$$\int \cot x\,dx = \int \cot y\,dy$$

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$\log(\sin x) - \log(\sin y) = \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

$$\Rightarrow u = \frac{\sin x}{\sin y}$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \cot y\,dy = \int \cot z\,dz$$

$$\log(\sin y) = \log(\sin z) + \log c_2$$

$$\log(\sin y) - \log(\sin z) = \log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$$\Rightarrow v = \frac{\sin y}{\sin z}$$

$$\Rightarrow \phi(u, v) = 0$$

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$$

5]. Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Soln.:

$$Pp + Qq = R \Rightarrow P = \sqrt{x},\quad Q = \sqrt{y},\quad R = \sqrt{z}$$

AE $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$

$2\sqrt{x} = 2\sqrt{y} + 2c_1$

$\sqrt{x} = \sqrt{y} + c_1$

$\sqrt{x} - \sqrt{y} = c_1$

$\Rightarrow u = \sqrt{x} - \sqrt{y}$

$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$

$2\sqrt{y} = 2\sqrt{z} + 2c_2$

$\sqrt{y} = \sqrt{z} + c_2$

$\sqrt{y} - \sqrt{z} = c_2$

$\Rightarrow v = \sqrt{y} - \sqrt{z}$

$\Rightarrow \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$

HW

1. $px + qy = \sqrt{z}$
2. $x^2 p + y^2 q = z^2$
3. $p - q = \log(x + y)$

Method of multipliers :

choose any 3 multiplier l, m, n which may be constant (or) functions of x, y, z, then

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$, then $l\,dx + m\,dy + n\,dz = 0$

Direct integration gives $u(x, y, z) = C_1$

Illy choose another set of 3 multipliers l', m' and n',

$v(x, y, z) = C_2$

$\Rightarrow \phi(u, v) = 0$

J. Solve $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

Soln. :

$x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2) \rightarrow (1)$

This eqn. is of the form,

$$Pp + Qq = R$$

where $P = x(z^2-y^2)$; $Q = y(x^2-z^2)$; $R = z(y^2-x^2)$

AE

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

Choosing x, y, z as lagrange's multipliers,

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2) + y^2(x^2-z^2) + z^2(y^2-x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + y^2z^2 - z^2x^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

ie, $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

$$x^2 + y^2 + z^2 = 2C_1$$

$\Rightarrow u = x^2 + y^2 + z^2$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as lagrange's multipliers

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(z^2-y^2) + \frac{1}{y}y(x^2-z^2) + \frac{1}{z}z(y^2-x^2)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

ie, $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrating, we get

$\log x + \log y + \log z = \log c_2$

$\log (xyz) = \log c_2$

$\Rightarrow c_2 = xyz$

$\Rightarrow v = xyz$

The Solution is, $\phi(u, v) = 0$

$\phi(x^2 + y^2 + z^2, xyz) = 0$

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2). Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Soln:

$(mz - ny)p + (nx - lz)q = ly - mx \rightarrow (1)$

This eqn. is of the form

$Pp + Qq = R$

where $P = (mz - ny)$; $Q = (nx - lz)$; $R = ly - mx$

AE $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$

choosing l, m, n as lagrange's multipliers,

$$\frac{l\,dx + m\,dy + n\,dz}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - lmz + lny - mnx}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

ie., $l\,dx + m\,dy + n\,dz = 0$

Integrating, we get

$$lx + my + nz = c_1$$

$$\Rightarrow u = lx + my + nz$$

choosing x, y, z as lagrange's multipliers, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - lyz + lyz - mxz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

ie., $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2 + y^2 + z^2 = 2c_1$$

$$\Rightarrow v = x^2 + y^2 + z^2$$

The solution is, $\phi(u, v) = 0$

$$\phi(lx + my + nz,\ x^2 + y^2 + z^2) = 0$$

3]. Solve $(3z-4y)p+(4x-2z)q = 2y-3x$

Soln.:

This eqn. is of the form

$$Pp+Qq=R$$

Here $P=3z-4y$; $Q=4x-2z$; $R=2y-3x$

AE

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$

choosing 2, 3, 4 as lagrange's multipliers,

$$\frac{2dx+3dy+4dz}{2(3z-4y)+3(4x-2z)+4(2y-3x)}$$

$$= \frac{2dx+3dy+4dz}{6z-8y+12x-6z+8y-12x}$$

$$= \frac{2dx+3dy+4dz}{0}$$

ie., $2dx+3dy+4dz=0$

Integrating, we get

$$2x+3y+4z=c_1$$

$$\Rightarrow u = 2x+3y+4z$$

choosing x, y, z as lagrange's multipliers,

$$\frac{xdx+ydy+zdz}{x(3z-4y)+y(4x-2z)+z(2y-3x)}$$

$$= \frac{xdx+ydy+zdz}{3xz-4xy+4xy-2yz+2yz-3xz}$$

$$= \frac{xdx+ydy+zdz}{0}$$

ie., $xdx+ydy+zdz=0$

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Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$x^2 + y^2 + z^2 = 2c_2$$

$$\Rightarrow v = x^2 + y^2 + z^2$$

The Solution is, $\phi(u, v) = 0$

$$\phi(2x + 3y + 4z,\ x^2 + y^2 + z^2) = 0$$