: Using Buckingham's $\pi$ theorem, show that the drag $F_D$ of a Supersonic aircraft is given by $F_D = \rho L^2 V^2 \phi(Re, M)$

where $Re = \rho V L / \mu$ = Reynolds number

$M = V/c$ = Mach number

$\rho$ = fluid density  $V$ = velocity of aircraft

$c$ = Sonic velocity $\sqrt{\frac{K}{\rho}}$

$K$ = bulk modulus of fluid

$L$ = Chord length

$L^2$ = wing area = chord x span

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Solution:

$$F_D = \rho L^2 V^2 \phi(Re, M)$$

$$F_D = \rho L^2 V^2 \phi\left[\frac{\rho V L}{\mu}, \frac{V}{\sqrt{\frac{K}{\rho}}}\right]$$

Therefore, drag is a function of $\rho$, $L$, $V$, $\mu$ and $K$

The functional relationship can be written as

$$F_D = f(\rho, L, V, \mu, K)$$

Again it can be written as

$$f_1 = (F_D \; \rho \; L \; V \; \mu \; K) \quad \text{———} (1)$$

Total number of variables $n = 6$

Fundamental parameters $m = 3$

$\pi$ terms $= n - m = 6 - 3 = 3$

Therefore the equation (1) can be written as

$$f_1(\pi_1 \ \pi_2 \ \pi_3) = 0 \qquad (2)$$

Each $\pi$ term has $m+1$ variables

Here $L, V$ and $\rho$ are selected as repeating Variables

$$\pi_1 = L^{a_1} \times V^{b_1} \times \rho^{c_1} \times F_D$$

$$\pi_2 = L^{a_2} \times V^{b_2} \times \rho^{c_2} \times \mu$$

$$\pi_3 = L^{a_3} \times V^{b_2} \times \rho^{c_2} \times K$$

Dimensions of each parameter are

Force $F_D = MLT^{-2}$

Diameter $L = L$

velocity $V = LT^{-1}$

density $\rho = ML^{-3}$

dynamic viscosity $\mu = ML^{-1}T^{-1}$

Bulk modulus $K = ML^{-1}T^{-2}$

$\pi_1$ - term

$$\pi_1 = L^{a_1} V^{b_1} \rho^{c_1} F_D$$

Substituting dimensions of each parameters / variables

$$M^0L^0T^0 = L^{a_1} \times (LT^{-1})^{b_1} \times (ML^{-3})^{c_1} \times MLT^{-2}$$

(16)

Comparing coefficient of each exponents on both sides

For M $\quad 0 = c_1 + 1 \quad$ —— (i)

L $\quad 0 = a_1 + b_1 - 3c_1 + 1 \quad$ —— (ii)

T $\quad 0 = -b_1 - 2 \quad$ —— (iii)

From (i) $\quad c_1 = -1$

From iii $\quad b_1 = -2$

Substituting values of $c_1$ & $b_1$ in eqn (ii)

$$0 = a_1 - 2 - 3(-1) + 1$$

$$a_1 = 2 - 3 - 1$$

$$a_1 = -2$$

Now $\pi_1 = L^{-2} V^{-2} \rho^{-1} F_D = \dfrac{F_D}{\rho V^2 L^2}$

$\pi_2$ - terms:

$$\pi_2 = L^{a_2} \times V^{b_2} \times \rho^{c_2} \times \mu$$

Substituting dimensions of each parameters / variables

$$M^0 L^0 T^0 = L^{a_2} \times (LT^{-1})^{b_2} \times (ML^{-3})^{c_2} \times ML^{-1}T^{-1}$$

Comparing coefficients of each exponent on both sides

For M $\quad 0 = c_2 + 1 \quad$ —— iv

L $\quad 0 = a_2 + b_2 - 3c_2 - 1 \quad$ —— v

T $\quad 0 = -b_2 - 1 \quad$ —— vi

From iv $c_2 = -1$ (17)

vi $b_2 = -1$

Substituting values of $c_2$ and $b_2$ in V

$$0 = a_2 - 1 - 3(-1) - 1$$

$$a_2 = 1 - 3 + 1 = -1$$

Now $\pi_2 = L^{-1} \times V^{-1} \times \rho^{-1} \times \mu$

$$= \frac{\mu}{\rho V L}$$

$\pi_3$ term $\pi_3 = L^{a_2} \times V^{b_2} \times \rho^{c_2} \times K$

Substituting dimensions of each Parameters Variables

$$M^0 L^0 T^0 = L^{a_1} \times (LT^{-1})(ML^{-3})^{c_1}$$

$$ML^{-1}T^{-2}$$

Comparing coefficient of each exponents on both sides

$M: \; 0 = c_3 + 1$ ——— vii

$L: \; 0 = a_3 + b_3 - 3c_3 - 1$ ——— viii

$T: \; 0 = -b_3 - 2$ ——— ix

From vii $c_1 = -1$

ix $b_1 = -2$

Substituting values of $c_3$ and $b_3$ in equation (viii)

$$0 = a_3 - 2 - 3(-1) - 1$$

$$a_3 = 2 - 3 + 1$$

$$a_3 = 0$$

Now $\pi_3 = L^0 V^{-2} \rho^{-1} K$

$$= \frac{K}{V^2 \rho}$$

Substituting $\pi_1$ $\pi_2$ and $\pi_3$ in Eqn (2)

$$f_1\left(\frac{F_D}{\rho V^2 L^2}, \frac{\mu}{\rho V L}, \frac{K}{V^2 \rho}\right) = 0$$

$$f_1 = \frac{F_D}{\rho V^2 L^2} f\left(\frac{\mu}{\rho V L} \ \frac{K}{V^2 \rho}\right)$$

$$F_D = \rho V^2 L^2 \varphi\left[\frac{\rho V L}{\mu}, \frac{V^2}{K/\rho}\right]$$

Taking Square root for $\frac{V^2}{K/\rho}$ only we get

$$F_D = \rho V^2 L^2 \phi\left(R_e, \frac{V}{\sqrt{K/\rho}}\right)$$

$$\boxed{F_D = \rho V^2 L^2 \phi\,(R_e, M)}$$

Unit II
16 Mark Qns

1. Consider force F acting on the propeller of an aircraft which depends upon the Variable $V, \rho, \mu, D$ and $N$. Desire the non-dimensional functional form

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{VD\rho}{\mu}\right) \frac{ND}{V}$$

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Solution:

The variables involved in analysis are $F, V, \rho, \mu, D$ and $N$

The dimensions of each variables are

Force $F = MLT^{-2}$

Velocity $V = LT^{-1}$

Density $\rho = ML^{-3}$

Dynamic viscosity $\mu = ML^{-1}T^{-1}$

diameter $D = L$

Speed $N = T^{-1}$

The functional relationship can be written as

$$F = f(V, \rho, \mu, D, N) \quad \text{—(1)}$$

Again it can be written as

$$f_1(F, V, \rho, \mu, D, N) = 0$$

$f_1(F, U, \rho, \mu, D, N) = 0$ ——— (2)

The total number of variables $n = 6$

Number of fundamental variables $m = 3$

$\therefore$ The number of $\pi$ terms $= n - m$

$= 6 - 3$

$= 3.$

So, the functional equation

$f_1(\pi_1, \pi_2, \pi_3) = 0$ ——— (3)

Each $\pi$-terms has $m+1$ variables

$\therefore \pi_1 = D^{a_1} \times U^{b_1} \times \rho^{c_1} \times F$

$\pi_2 = D^{a_2} \times U^{b_2} \times \rho^{c_2} \times \mu$

$\pi_3 = D^{a_3} \times U^{b_3} \times \rho^{c_3} \times N$

$\pi_1$ - term

$\pi_1 = D^{a_1} \times U^{b_1} \times \rho^{c_1} \times F$

Now, the Dimensionless equation becomes

$M^0 L^0 T^0 = L^{a_1} \times (LT^{-1})^{b_1} \times (ML^{-3})^{c_1} \times MLT^{-2}$

Comparing exponents coefficients on both sides

For M: $0 = c_1 + 1$ ——— (i)

For L: $0 = a_1 + b_1 - 3c_1 + 1$ ——— (ii)

For T: $0 = -b_1 - 2$ ——— (iii)

From (i) $c_1 = -1.$

From (iii) $b_1 = -2$

From (ii) $a_1 = -b_1 + 3c_1 - 1.$

$$= -(-2) + 3(-1) - 1 = 2 - 3 - 1 = -2$$

$$\therefore \pi_1 = D^{-2} \times U^{-2} \times \rho^{-1} \times F$$

$$\pi_1 = \frac{F}{\rho U^2 D^2}$$

$\pi_2$ - term

$$\pi_2 = D^{a_2} \times U^{b_2} \times \rho^{c_2} \times \mu$$

Now, the dimension less equation becomes

$$M^0 L^0 T^0 = L^{a_2} \times (LT^{-1})^{b_2} \times (ML^{-3})^{c_2} \times ML^{-1}T^{-1}$$

Comparing exponents co-efficient on both sides

for $M: \; 0 = c_2 + 1$ —(iv)

$L = 0 = a_2 + b_2 - 3c_2 - 1$ —(v)

$T = 0 = -b_2 - 1$ —(vi)

from iv $c_2 = -1$

vi $b_2 = -1$

v $a_2 = -b_2 + 3c_2 + 1 = -(-1) + 3(-1) + 1$

$= 1 - 3 + 1 = -1.$

$$\pi_2 = D^{-1} \times U^{-1} \times \rho^{-1} \times \mu$$

$$= \frac{\mu}{\rho U D}$$

$\pi_3$ – term

$$\pi_3 = D^{a_3} \times U^{b_3} \times \rho^{c_3} \times N$$

Now, the dimensionless equation becomes

$$M^0 L^0 T^0 = L^{a_3} \times (LT^{-1})^{b_3} \times (ML^{-3})^{c_3} \times T^{-1}$$

Comparing exponents Coefficient on both sides

For M $\quad 0 = c_3$ ——— vii

L $\quad 0 = a_3 + b_3 - 3c_3$ — viii

T $\quad 0 = -b_3 - 1$ ——— ix

From vii $\quad c_3 = 0$

ix $\quad b_3 = -1$

viii $\quad a_3 = -b_3 + 3c_3$

$= -(-1) + 3 \times 0$

$= 1.$

$$\pi_3 = D^1 U^{-1} \rho^0 N = \frac{ND}{U}$$

Substituting values $\pi_1$, $\pi_2$ & $\pi_3$ in eqn (3)

$$f_1\left(\frac{F}{\rho U^2 D^2}, \frac{\mu}{\rho U D}, \frac{ND}{U}\right) = 0$$

$$\therefore \frac{F}{\rho U^2 D^2} = f\left(\frac{\mu}{\rho U D} \quad \frac{ND}{U}\right)$$

$$\frac{F}{\rho U^2 D^2} = f\left(\frac{UD\rho}{\mu}, \frac{ND}{U}\right)$$

$\left[\because \text{Dimensionless parameter } \frac{\mu}{\rho UD} = \frac{UD\rho}{\mu}\right]$