Unit - II

Combinatorics

Principle of mathematical Induction:

Let $P(n)$ be a statement or proposition involving for all positive integers n.

Step 1: $P(1)$ is true.

Step 2: Assume that $P(K)$ is true

Step 3: we have to prove that $P(K+1)$ is true.

Problems:

1. prove that $2+2^2+\dots+2^n = 2^{n+1}-2$ by mathematical induction.

$P(n): 2+2^2+\dots+2^n = 2^{n+1}-2$

Step 1: $P(1) : 2^1 = 2^{1+1}-2$

$= 2^2-2$

$= 4-2$

$2 = 2$

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(K): 2^1+2^2+\dots+2^K = 2^{K+1}-2$ is true.

Step 3: To prove $P(K+1)$ is true.

Now

$P(K+1) = 2^1+2^2+\dots+2^K+2^{K+1}$

$= 2^{K+1}-2+2^{K+1}$

$= 2^1 . 2^{K+1}-2$

$= 2^{(K+1)+1}-2$

$\therefore$ $P(K+1)$ is true.

$\therefore$ $P(n): 2+2^2+\dots+2^n = 2^{n+1}-2$ is true, for all n.

2]. prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ by mathematical Induction

Let $P(n) : 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: $P(1)$:

LHS : $1^2 = 1$

RHS : $\frac{1(1+1)(2+1)}{6} = \frac{2(3)}{6} = 1$

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(K)$: $1^2 + 2^2 + 3^2 + \dots + K^2 = \frac{K(K+1)(2K+1)}{6}$ is true.

Step 3: To prove $P(K+1)$ is true.

Now, $P(K+1) = 1^2 + 2^2 + 3^2 + \dots + K^2 + (K+1)^2$

$$= \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= \frac{K(K+1)(2K+1) + 6(K+1)^2}{6}$$

$$= \frac{(K+1)[K(2K+1) + 6(K+1)]}{6}$$

$$= \frac{(K+1)[2K^2 + K + 6K + 6]}{6}$$

$$= \frac{(K+1)[2K^2 + 7K + 6]}{6}$$

$$= \frac{(K+1)[2K^2 + 4K + 3K + 6]}{6}$$

$$= \frac{(K+1)}{6}[2K(K+2) + 3(K+2)]$$

$$= \frac{(K+1)}{6}[(K+2)(2K+3)]$$

$$= \frac{(K+1)((K+1)+1)(2K+2+1)}{6}$$

$$= \frac{(K+1)\,[(K+1)+1]\,[2(K+1)+1]}{6}$$

$\therefore$ $P(K+1)$ is true.

Hence $P(n)$: $\sum_{K=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$ is true, $\forall n$.

3] Prove that $1+2+\dots+n = \frac{n(n+1)}{2}$

Let $P(n)$: $1+2+\dots+n = \frac{n(n+1)}{2}$

Step 1:

$P(1)$: LHS $= 1$

RHS $= \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(K)$: $1+2+\dots+K = \frac{K(K+1)}{2}$ is true

Step 3: To prove $P(K+1)$ is true.

Now, $P(K+1) = 1+2+\dots+K+K+1$

$$= \frac{K(K+1)}{2} + K+1$$

$$= \frac{K(K+1)+2(K+1)}{2}$$

$$= \frac{(K+1)(K+2)}{2}$$

$$= \frac{(K+1)\,[(K+1)+1]}{2}$$

$\therefore$ $P(K+1)$ is true.

$\therefore$ $P(n)$: $1+2+\dots+n = \frac{n(n+1)}{2}$ is true, for all n.

4] Show that $8^n - 3^n$ is a multiple of 5.

Let $P(n)$: $8^n - 3^n$ be a multiple of 5.

Step 1: $P(1): 8^1 - 3^1 = 5$ is a multiple of 5

$\therefore$ $P(1)$ is true.

Step 2: Assume that

$P(K): 8^K - 3^K$ is a multiple of 5, is true.

ie, $8^K - 3^K = 5m \Rightarrow 8^K = 3^K + 5m$

Step 3: To prove $P(K+1)$ is true.

Now $P(K+1) = 8^{K+1} - 3^{K+1}$

$= 8^K . 8^1 - 3^K . 3^1$

$= (3^K + 5m) \cdot 8 - 3^K . 3$

$= 5m . 8 + 3^K . 8 - 3^K . 3$

$= 5m . 8 + 3^K (8-3)$

$= 5m . 8 + 3^K . 5$

$= 5 [8m + 3^K]$ which is a multiple of 5, $\forall n$

$\therefore$ $P(K+1)$ is true.

$\therefore$ $P(n)$: $8^n - 3^n$ is a multiple of 5, $\forall n$

5]. Use mathematical induction, prove that $3^n + 7^n - 2$ is divisible by 8, for $n \geq 1$

Let $P(n): 3^n + 7^n - 2$ be divisible by 8, for $n \geq 1$

Step 1: $P(1)$: $3^1 + 7^1 - 2 = 3 + 7 - 2$

$= 8$ is divisible by 8, $n \geq 1$

Step 2: Assume that $P(K)$: $3^K + 7^K - 2$ is divisible by 8, $n \geq 1$ is true.

ie., $P(K) = \dfrac{3^K + 7^K - 2}{8} = m$

$\Rightarrow 3^K + 7^K - 2 = 8m$

$\Rightarrow 3^K = 8m + 2 - 7^K$

Step 3: To prove $P(K+1)$ is true.

Now $P(K+1) = 3^{K+1} + 7^{K+1} - 2$

$= 3^K \cdot 3 + 7^K \cdot 7 - 2$

$= (8m + 2 - 7^K) \cdot 3 + 7^K \cdot 7 - 2$

$= 24m + 6 - 3 \cdot 7^K + 7^K \cdot 7 - 2$

$= 24m + 4 + 7^K (7 - 3)$

$= 24m + 4 + 7^K \cdot 4$

$= 4(6m + 1 + 7^K)$, $m \geq 1, K \geq 1$

$\because 6m + 1 + 7^K$ is an even number.

$\Rightarrow 4(6m + 1 + 7^K)$ is divisible by 8.

$\therefore P(K+1)$ is true.

$\therefore P(n) = 3^n + 7^n - 2$ is divisible by 8, $n \geq 1$.

6] Prove that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let $P(n): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1: $P(1)$

LHS: $\frac{1}{1 \cdot 2} = \frac{1}{2}$

RHS: $\frac{1}{1+1} = \frac{1}{2}$

Step 2: Assume that

$P(K): \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{K(K+1)} = \frac{K}{K+1}$

Step 3: To prove $P(K+1)$ is true.

Now, $P(K+1) = \frac{1}{1 \cdot 2} + \dots + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)}$

$= \frac{K}{K(K+1)} + \frac{1}{(K+1)(K+2)}$

$= \frac{K(K+2)+1}{(K+1)(K+2)} = \frac{K^2 + 2K + 1}{(K+1)(K+2)}$

$= \frac{(K+1)(K+1)}{(K+1)(K+2)} = \frac{K+1}{(K+1)+1}$ is true, $\forall n$

① $n < 2^n$

② $n^3 + 2n$ is divisible by 3