Transforms of simple functions:

I. Find the fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$$ and hence prove that

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{3\pi}{16}$$

Soln:

Given $f(x) = \begin{cases} 1-x^2, & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\,(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^1 (1-x^2)\cos sx\, dx + i(0) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + \cancel{(-2)} \right.$$
$$\left. (-2)\left(-\frac{\sin sx}{s^3}\right) - 0 \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\frac{\sin sx}{s} - \frac{2x\cos sx}{s^2} + \frac{2\sin sx}{s^3} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right) - 0 \right] \cancel{2\sin}$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

By using IFT,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right] (\cos sx - i \sin sx) \, ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - i\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx \, ds\right]$$

$$= \frac{4}{\pi}\left[\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - i(0)\right]$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds = \frac{\pi}{4} f(x)$$

$$\because f(x) = 1 - x^2$$

$$f(1/2) = 1 - 1/4 = 3/4$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos s/2 \, ds = \frac{3\pi}{16}$$

1. Find the F.T. of $e^{-a|x|}$ and hence deduce that i). $F[x e^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$

Soln. :

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

Now,

$$F[e^{-a|x|}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\,(\cos sx + i\sin sx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax} \cos sx\, dx \qquad \because |x| = x,\ (0,\infty)$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+s^2}$$

i). $F[x e^{-a|x|}] = -i\frac{d}{ds}F(s)$

$$= -i\frac{d}{ds} F[e^{-a|x|}]$$

$$= -i\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+s^2}\right]$$

$$= -ia\sqrt{\frac{2}{\pi}}\left[\frac{0-2s}{(s^2+a^2)^2}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\ \frac{2as}{(s^2+a^2)^2}$$