Harmonic Analysis

The process of finding the fourier series for a function given by numerical values is known as harmonic analysis.

Formula:

$\Rightarrow$ π form or T form or Degree form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right)$$

$$a_n = 2\left(\frac{\Sigma y \cos nx}{N}\right) \Rightarrow a_1 = 2\left(\frac{\Sigma y \cos x}{N}\right)$$

$$a_2 = 2\left(\frac{\Sigma y \cos 2x}{N}\right) \text{ and so on.}$$

$$b_n = 2\left(\frac{\Sigma y \sin nx}{N}\right) \Rightarrow b_1 = 2\left(\frac{\Sigma y \sin x}{N}\right)$$

$$b_2 = 2\left(\frac{\Sigma y \sin 2x}{N}\right) \text{ and so on.}$$

$\Rightarrow$ l form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right); \quad a_1 = 2\,\frac{\Sigma y \cos\left(\frac{\pi x}{l}\right)}{N}$$

$$a_2 = 2\,\frac{\Sigma y \cos\left(\frac{2\pi x}{l}\right)}{N}$$

and so on

$$b_1 = 2\left(\frac{\Sigma y \sin \frac{\pi x}{l}}{N}\right)$$

$$b_2 = 2\left(\frac{\Sigma y \sin \frac{2\pi x}{l}}{N}\right)$$

I. Find the Fourier series expansion defined in $(0, T)$ by means of the table of values given below. Find the series upto the 2nd harmonic.

t sec.	0	T/6	T/3	T/2	2T/3	5T/6	T
A-temp	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Soln.

t sec	0	T/6	T/3	T/2	2T/3	5T/6	~~T~~
A-temp	1.98	1.30	1.05	1.30	-0.88	-0.25	~~1.98~~

N = Number of terms = 6

$T = 2\pi = 360^\circ$

$2l = 2\pi \Rightarrow l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

upto 2nd harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x \quad \to (1)$$

Subs. $T = 2\pi$ in the table,

t sec	0	60°	120°	180°	240°	300°
A-temp	1.98	1.30	1.05	1.30	-0.88	-0.25

$$T/6 = \frac{2\pi}{6} = \frac{360}{6} = 60^\circ$$

$$T/3 = \frac{2\pi}{3} = \frac{360}{3} = 120^\circ$$

$\vdots$

$$\frac{5T}{6} = \cancel{\frac{1800}{3}}\ \frac{5(2\pi)}{3} = \frac{10\pi}{6} = \frac{1800}{6} = 300^\circ$$

HARMONIC ANALYSIS

x	$y=f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.98	1.98	1.98	0	0
60°	1.30	0.65	−0.65	1.126	1.126
120°	1.05	−0.525	−0.525	0.909	−0.909
180°	1.30	−1.3	1.3	0	0
240°	−0.88	0.44	0.44	0.762	−0.762
300°	−0.25	−0.125	0.125	0.217	0.217
	$\Sigma y = 4.5$	$\Sigma y\cos x = 1.12$	$\Sigma y\cos 2x = 2.67$	$\Sigma y\sin x = 3.014$	$\Sigma y\sin 2x = -0.328$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{4.5}{6}\right), \quad a_0 = 1.5$$

$$a_1 = 2\left(\frac{\Sigma y\cos x}{N}\right) = 2\left(\frac{1.12}{6}\right), \quad a_1 = 0.373$$

$$a_2 = 2\left(\frac{\Sigma y\cos 2x}{N}\right) = 2\left(\frac{2.67}{6}\right), \quad a_2 = 0.89$$

$$b_1 = 2\left(\frac{\Sigma y\sin x}{N}\right) = 2\left(\frac{3.014}{6}\right), \quad b_1 = 1.005$$

$$b_2 = 2\left(\frac{\Sigma y\sin 2x}{N}\right) = 2\left(\frac{-0.328}{6}\right), \quad b_2 = -0.109$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

$$f(x) = 0.75 + 0.373\cos x + 0.89\cos 2x + 1.005\sin x - 0.109\sin 2x$$

2]. Find the Fourier Series upto second harmonic for the following data.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Soln.

$N = 6$ (omit the last value)

$2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

x	$y = f(x)$	$y \cdot \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.0	1	1	0	0
60°	1.4	0.7	−0.7	1.212	1.212
120°	1.9	−0.95	−0.95	1.645	−1.645
180°	1.7	−1.7	1.7	0	0
240°	1.5	−0.75	−0.75	−1.299	1.299
300°	1.2	0.6	−0.6	−1.039	−1.039
	Σy 8.7	$\Sigma y \cos x = -1.1$	$\Sigma y \cos 2x = -0.3$	$\Sigma y \sin x = 0.5196$	$\Sigma y \sin 2x = -0.1732$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{8.7}{6}\right) = 2.9$$

$$a_1 = 2\left(\frac{\Sigma y \cos x}{N}\right) = 2\left(\frac{-1.1}{6}\right) = -0.37$$

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$$a_2 = 2\left(\frac{\Sigma y \cos 2x}{N}\right) = 2\left(\frac{-0.3}{6}\right) = -0.1$$

$$b_1 = 2\left(\frac{\Sigma y \sin x}{N}\right) = 2\left(\frac{0.5196}{6}\right) = 0.17$$

$$b_2 = 2\left(\frac{\Sigma y \sin 2x}{N}\right) = 2\left(\frac{-0.1732}{6}\right) = -0.06$$

$$\therefore f(x) = 1.45 - 0.37 \cos x - 0.1 \cos 2x + 0.17 \sin x - 0.06 \sin 2x$$

3]. Find the Fourier series as far as the second harmonic to represent the function given in the following data

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Soln.

$N = 6$

$2l = 6 \Rightarrow l = 3$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$= \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + a_2 \cos\left(\frac{2\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) + b_2 \sin\left(\frac{2\pi x}{3}\right)$$

x	$y=f(x)$	$y\cos\left(\frac{\pi x}{3}\right)$	$y\cos\left(\frac{2\pi x}{3}\right)$	$y\sin\left(\frac{\pi x}{3}\right)$	$y\sin\left(\frac{2\pi x}{3}\right)$
0	0	9	9	0	0
1	18	9	−9	15.7	15.6
2	24	−12	−24	20.9	0
3	28	−28	28	0	0
4	26	−13	−13	−22.6	22.6
5	20	10	−10	−17.4	−17.4
	$\Sigma y = 125$	$\Sigma y\cos(\pi x/3) = -25$	$\Sigma y\cos\left(\frac{2\pi x}{3}\right) = -19$	$\Sigma y\sin\left(\frac{\pi x}{3}\right) = -3.4$	$\Sigma y\sin\left(\frac{2\pi x}{3}\right) = 20.8$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{125}{6}\right) = 41.66$$

$$a_1 = 2\left(\frac{\Sigma y\cos\left(\frac{\pi x}{3}\right)}{N}\right) = 2\left(\frac{-25}{6}\right) = -8.33$$

$$a_2 = 2\left(\frac{\Sigma y\cos\left(\frac{2\pi x}{3}\right)}{N}\right) = 2\left(\frac{-19}{6}\right) = -6.33$$

$$b_1 = 2\left(\frac{\Sigma y\sin\left(\frac{\pi x}{3}\right)}{N}\right) = 2\left(\frac{-3.4}{6}\right) = -1.13$$

$$b_2 = 2\left(\frac{\Sigma y\sin\left(\frac{2\pi x}{3}\right)}{N}\right) = 2\left(\frac{20.8}{6}\right) = 0.009$$

$$\therefore f(x) = 20.83 - 8.33\cos\left(\frac{\pi x}{3}\right) - 6.33\cos\left(\frac{2\pi x}{3}\right) - 1.13\sin\left(\frac{\pi x}{3}\right) + 0.009\sin\left(\frac{2\pi x}{3}\right)$$