2. Fourier Transform

Fourier Transform Pair :

The fourier transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \rightarrow ①$$

→ complex no

Complex fourier transform of $f(x)$

$$e^{isx} = \cos sx + i \sin sx$$

Then the function $f(x)$ is the Inverse Fourier transform of $F(s)$ is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \longrightarrow ②.$$

$$e^{-isx} = \cos sx - i \sin sx$$

The above eqns ① and ② are jointly called Fourier Transform pair,

$$f(x) = F^{-1}[F(s)] = F^{-1}[F[f(x)]]$$

1. Find the fourier transform of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

and hence deduce that $\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$

Fourier Transform is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1(\cos sx + i \underbrace{\sin sx}_{\text{odd}})\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx \quad = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - \frac{\sin 0}{s} \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin sa}{s} \right)$$

Inverse fourier transform is $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sin sa}{s} (\cos sx - i \sin sx) ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \left[\int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos sx \, ds - i \int_{-\infty}^{\infty} \frac{\sin sa}{s} \sin sx \, ds \right]$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos sx \, ds$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin sa}{s} \cos sx \, ds$$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin sa}{s} \cos sx \, ds$$

Put $x=0$, $a=1$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin s}{s} ds$$ put $s=t$, $ds=dt$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin t}{t} dt$$

② Find the FT of $f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \quad \& \quad a < x < \infty \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x [\cos sx + i \sin sx] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x \cos sx \, dx + i \int_{-a}^{a} x \sin sx \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2i \int_{0}^{a} x \sin sx \, dx \right]$$

$$= i\sqrt{\frac{2}{\pi}} \left[x \left(\frac{-\cos sx}{s} \right) - 1 \left(\frac{-\sin sx}{s^2} \right) + 0 \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, i \left[\left(-\frac{a \cos sa}{s} + \frac{\sin sa}{s^2} \right) - 0 \right]$$

$$= \sqrt{\frac{2}{\pi}}\, i \left[\frac{\sin sa - sa \cos sa}{s^2} \right]$$

③ Find the fourier transform of the function

$f(x) = \begin{cases} 1 - x^2 & , |x| < 1 \\ 0 & , |x| > 1 \end{cases}$ & Hence deduce that

$$\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cdot \cos \frac{s}{2} \, ds = \frac{3\pi}{16}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) e^{isx} \, dx.$$

Show that the fourier transform of

$$f(x) = \begin{cases} a^2 - x^2 & , |x| < a \\ 0 & , |x| > a > 0 \end{cases}$$ and hence find that

$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$. Hence deduce that

$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}$. using P.I show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$$

$$f(x) = \begin{cases} a^2 - x^2 & , -a < x < a \\ 0 & , -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} (a^2 - x^2)\cos sx\, dx + i\int_{-a}^{a} (a^2 - x^2)\sin sx\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a^2 - x^2)\cos sx\, dx + i(0)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]$$

put $a = 1$ $\quad F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

i) Using Inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right] \left[\cos sx - i \sin sx\right] ds$$

$$= \frac{2}{\pi} \cdot 2\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds$$

Put $x=0$ $\quad f(0) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}.$$

ii) Using Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left\{2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)\right\}^2 ds$$

$$2\int_{0}^{1} (1+x^4-2x^2)\, dx = \frac{16}{\pi} \int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi} \int \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15+3-10}{15}\right] = \frac{16}{15}$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}\left(\frac{\pi}{16}\right)$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

Show that the fourier transform of $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 > 0 \end{cases}$

and hence find that $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$. Hence deduce

that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}$. Using Parseval's Identity show

that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

$$f(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0, & -\infty < x < 1 \ \& \ 1 < x < \infty \end{cases}$$

By Fourier Transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)(\cos sx + i \sin sx) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) \cos sx \, dx \quad = \sqrt{\frac{2}{\pi}} \int_0^1 (1 - x^2) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1 - x^2) \frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

By Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right] \right] (\cos sx - i \sin sx) ds$$

$$= \frac{4}{\pi} \int_0^\infty \left[\frac{\sin s - s\cos s}{s^3} \right] \cos sx \, ds$$

Put $x = 0$

$$\frac{\pi}{4} f(0) = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right) ds$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right) ds = \frac{\pi}{4}$$

Replace s by t

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}.$$

By Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds$$

$$2\int_0^1 (1+x^4-2x^2)\,dx = \frac{4.2.2}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left[x+\frac{x^5}{5}-\frac{2x^3}{3}\right]_0^1 = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left[1+\frac{1}{5}-\frac{2}{3}\right] = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left[\frac{15+3-10}{15}\right] = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left(\frac{8}{15}\right) = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

Replace s by t

$$\frac{\pi}{16} = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 ds$$