Parseval's Identity (or) Rayleigh's Theorem

Let $F(s)$ be the Fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Result :- $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Self Reciprocal Function:

If the fourier transform of $f(x)$ is equal to $F(s)$, then $f(x)$ is said to be reciprocal function under Fourier transform

i.e) $F[f(x)] = F(s)$

eg: $F[e^{-x^2/2}] = e^{-s^2/2}$.

Show that the function $e^{-x^2/2}$ is self reciprocal under Fourier Transform.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2} + isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2 - 2isx)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}[x^2 - 2isx + (is)^2 - (is)^2]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{(x-is)^2}{2}\right] + \frac{(is)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{(is)^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} dx$$ Put $t = \frac{x-is}{\sqrt{2}} \Rightarrow dt = \frac{dx}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} \cdot \sqrt{2}\, dt$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} e^{-s^2/2} \cdot \sqrt{\pi}$$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ ⇓ Gaussian Integral

$$= e^{-s^2/2}.$$

$\therefore$ The fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self reciprocal under fourier transform.

Find the F.T of $f(x) = \begin{cases} a-|x| & , |x|<a \\ 0 & , |x|>a>0. \end{cases}$ and

deduce that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt$

$$f(x) = \begin{cases} a-|x| & , -a<x<a \\ 0 & , -\infty<x<-a, \ a<x<\infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) [\cos sx + i \sin sx] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{a} (a-x) (\cos sx + i \sin sx) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (a-x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(a-x)\frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s^2}\right]_0^a = -\sqrt{\frac{2}{\pi}}\left[\frac{\cos sa}{s^2} - \frac{\cos 0}{s^2}\right.$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right) = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2 sa/2}{s^2}\right]$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos\theta = 1 - 2\sin^2\theta/2$

i) using IFT

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2 sa/2}{s^2}\right](\cos sx - i\sin sx)\, dx$$

$$= \frac{2.2}{\pi}\int_0^{\infty}\frac{1}{s^2}\cdot\sin^2\frac{sa}{2}\,[\cos sx]\, ds$$

Put $x=0$, $a=2$

$$f(0) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 s}{s^2}\, ds$$

$$\therefore \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(2-|0|)$$

$$= \frac{\pi}{2}.$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

(ii) Parseval's Identity :-

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-2}^{2} [2-|x|]^2 dx = \int_{-\infty}^{\infty} 4 \cdot \frac{2}{\pi} \frac{\sin^4 \frac{sa}{2}}{s^4} ds$$

$$2\int_{0}^{2} (2-x)^2 dx = \frac{8}{\pi} \cdot 2 \int_{0}^{\infty} \left[\frac{\sin \frac{sa}{2}}{s}\right]^4 ds$$

$a = 2$

$$2\left[\frac{(2-x)^3}{3}\right]_0^2 = \frac{16}{\pi}\int_0^{\infty}\left[\frac{\sin s}{s}\right]^4$$

$$\left(\frac{-2}{3}\right)(-8)\frac{\pi}{16} = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$$

$$\frac{\pi}{3} = \int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$$

Find the fourier Transform of $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| > 0 \end{cases}$

and deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ and $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

By Fourier Transform,

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx) dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s^2} + \frac{\cos 0}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$1 - \cos 2\theta = 2\sin^2\theta$$
$$1 - \cos\theta = 2\sin^2\theta/2$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 s/2}{s^2} \right]$$

By Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left(\frac{2\sin^2 s/2}{s^2} \right) \right) (\cos sx - i \sin sx) \, ds$$

$$= \frac{2 \cdot 2}{\pi} \int_0^{\infty} \left(\frac{\sin^2 s/2}{s^2} \right) \cos sx \, ds$$

Put $x = 0$ and $s/2 = t$ $\quad s = 2t \quad ds = 2dt$

$$f(0) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin^2 t}{(2t)^2} \right) 2dt$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 t}{4t^2} \cdot 2dt \quad \Rightarrow 1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt$$

$$\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

By Parsevals Identity :-

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-|x|)^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 s/2}{s^2}\right]\right]^2 ds$$

$$2\int_{0}^{1} (1-x)^2 dx = \frac{16}{\pi} \int_{0}^{\infty} \left(\frac{\sin^4 s/2}{s^4}\right) ds$$

Put $s/2 = t$ $\quad 2t = s \quad ds = 2dt$

$$2\int_{0}^{1} (1+x^2-2x)\, dx = \frac{16}{\pi} \int_{0}^{\infty} \left(\frac{\sin^4 t}{(2t)^4}\right) 2dt$$

$$2\left[x + \frac{x^3}{3} - \frac{2x^2}{2}\right]_0^1 = \frac{16}{\pi} \int_{0}^{\infty} \frac{\sin^4 t}{16t^4} \cdot 2dt$$

$$(\cancel{1} + \frac{1}{3} - \cancel{1}) - (0) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin^4 t}{t^4} dt$$

$$\frac{\pi}{3} = \int_{0}^{\infty} \frac{\sin^4 t}{t^4} dt$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$