Harmonic Analysis

The Process of finding the fourier series for a function given by numerical values is known as harmonic analysis.

Formula:

$\Rightarrow$ π form or T form or Degree form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right)$$

$$a_n = 2\left(\frac{\Sigma y \cos nx}{N}\right) \Rightarrow a_1 = 2\left(\frac{\Sigma y \cos x}{N}\right)$$

$$a_2 = 2\left(\frac{\Sigma y \cos 2x}{N}\right) \text{ and so on.}$$

$$b_n = 2\left(\frac{\Sigma y \sin nx}{N}\right) \Rightarrow b_1 = 2\left(\frac{\Sigma y \sin x}{N}\right)$$

$$b_2 = 2\left(\frac{\Sigma y \sin 2x}{N}\right) \text{ and so on.}$$

$\Rightarrow$ l form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) \; ; \; a_n = 2\,\frac{\Sigma y \cos\left(\frac{n\pi x}{l}\right)}{N} \; ; \; b_n = \frac{\Sigma y \sin \frac{n\pi x}{l}}{N}$$

$$a_1 = 2\,\frac{\Sigma y \cos \frac{\pi x}{l}}{N} \qquad b_1 = 2\,\frac{\Sigma y \sin \frac{\pi x}{l}}{N}$$

$$a_2 = 2\,\frac{\Sigma y \cos \frac{2\pi x}{l}}{N} \qquad b_2 = 2\,\frac{\Sigma y \sin \frac{2\pi x}{l}}{N}$$

1. Find the fourier series expansion defined in $(0, T)$ by means of the table values given below. find the series upto the 2^{nd} harmonic.

t sec	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A temp.	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Soln :

t sec	0	$\frac{T}{6}$	$T/3$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$
A temp	1.98	1.30	1.05	1.30	-0.88	- 0.25

$N =$ Number of terms $= 6$

$T = 2\pi = 360^\circ$

$2l = 2\pi \Rightarrow l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Upto 2^{nd} harmonic.

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x \quad \rightarrow (1)$$

Sub $T = 2\pi$ in the table

t sec	0	60°	120°	180°	240°	300°
A temp	1.98	1.30	1.05	1.30	-0.88	-0.25

x	$y=f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.98	1.98	1.98	0	0
60°	1.30	0.65	−0.65	1.126	1.126
120°	1.05	−0.525	−0.525	0.909	−0.909
180°	1.30	−1.3	1.3	0	0
240°	−0.88	0.44	0.44	0.762	−0.762
300°	−0.25	−0.125	0.125	0.217	0.217
	$\Sigma y = 4.5$	$\Sigma y\cos x = 1.12$	$\Sigma y\cos 2x = 2.67$	$\Sigma y\sin x = 3.014$	$\Sigma y\sin 2x = -0.328$

$$a_0 = 2\frac{\Sigma y}{N} = 2\left(\frac{4.5}{6}\right)$$

$$a_0 = 1.5$$

$$a_1 = 2\left(\frac{\Sigma y\cos x}{N}\right) = 2\left(\frac{1.12}{6}\right)$$

$$a_1 = 0.373$$

$$a_2 = 2\left(\frac{\Sigma y\cos 2x}{N}\right) = 2\left(\frac{2.67}{6}\right)$$

$$a_2 = 0.89$$

$$b_1 = 2\left(\frac{\Sigma y\sin x}{N}\right) = 2\left(\frac{3.014}{6}\right)$$

$$b_1 = 1.005$$

$$b_2 = 2\left(\frac{\Sigma y\sin 2x}{N}\right) = 2\left(\frac{-0.328}{6}\right)$$

$$b_2 = -0.109$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

$$= 0.75 + 0.373\cos x + 0.89\cos 2x + 1.005\sin x - 0.109\sin 2x$$

2. Find the fourier series upto second harmonic for the following data

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0.

Soln $N=6$

$2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x.$$

x	$y=f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.0	1	1	0	0
60° 0.5	1.4	0.7	-0.7 0.866	1.212	1.212
120° 0.5	1.9	-0.95	-0.95 0.866	1.645	-1.645
180° -1	1.7	-1.7	1.7 0	0	0
240° -0.5	1.5	-0.75	-0.75 -0.866	-1.299	1.299
300° 0.5	1.2	0.6	-0.6 -0.866	-1.039	-1.039
	Σy = 8.7	$\Sigma y\cos x$ = -1.1	$\Sigma y\cos 2x$ = 0.3	$\Sigma y\sin x$ = 0.5196	$\Sigma y\sin 2x$ = -0.1732

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{8.7}{6}\right) = 2.9$$

$$a_1 = 2\left(\frac{\Sigma y\cos x}{N}\right) = 2\left(\frac{-1.1}{6}\right) = -0.37$$

$$a_2 = 2\left(\frac{\Sigma y\cos 2x}{N}\right) = 2\left(\frac{-0.3}{6}\right) = -0.1$$

$$b_1 = 2\left(\frac{\Sigma y\sin x}{N}\right) = 2\left(\frac{0.5196}{6}\right) = 0.17$$

$$b_2 = 2\left(\frac{\Sigma y\sin 2x}{N}\right) = 2\left(\frac{-0.1732}{6}\right) = -0.06$$

$\therefore f(x) = 1.45 - 0.37\cos x - 0.1\cos 2x + 0.17\sin x - 0.06\sin 2x$

3. find the fourier series as far as the second harmonic to represent the function given in the following data

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Soln: $N = 6$

$2l = 6 \Rightarrow \boxed{l = 3}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{3}\right)$$

$$= \frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{3}\right) + a_2\cos\left(\frac{2\pi x}{3}\right) + b_1\sin\left(\frac{\pi x}{3}\right)$$

$$+ b_2\sin\left(\frac{2\pi x}{3}\right)$$

x	$y=f(x)$	$y\cos\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{n\pi}{3}$	$y\cos\frac{2\pi x}{3}$
0	0	9	9	0	0
1	18	9	-9	15.7	15.6
2	24	-12	-24	20.9	0
3	28	-28	28	0	0
4	26	-13	-13	-22.6	22.6
5	20	10	-10	-17.4	-17.4
	Σy $= 125$	$\Sigma y\cos\left(\frac{\pi x}{3}\right)$ $= -25$	$\Sigma y\cos\left(\frac{2\pi x}{3}\right)$ $= -19$	$\Sigma y\sin\left(\frac{n\pi}{3}\right)$ $= -3.4$	$\Sigma y\sin\left(\frac{2\pi x}{3}\right)$ $= 20.8$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\left(\frac{125}{6}\right) = 41.66$$

$$a_1 = 2\left(\frac{\Sigma y\cos(n\pi/3)}{N}\right) = 2\left(\frac{-25}{6}\right) = -8.33$$

$$a_2 = 2\left(\frac{\Sigma y\cos\left(\frac{2\pi x}{3}\right)}{N}\right) = 2\left(\frac{-19}{6}\right) = -6.33$$

$$b_1 = 2\left(\frac{\Sigma y\sin\left(\frac{\pi x}{3}\right)}{N}\right) = 2\left(\frac{-3.4}{6}\right) = -1.13$$

$$b_2 = 2\left(\frac{\Sigma y\sin\left(\frac{2\pi x}{3}\right)}{N}\right) = 2\left(\frac{20.8}{6}\right) = 0.009$$

$$\therefore f(x) = 20.83 - 8.33\cos\left(\frac{\pi x}{3}\right) - 6.33\cos\left(\frac{2\pi x}{3}\right)$$

$$-1.33\sin\left(\frac{\pi x}{3}\right) + 0.009\sin\left(\frac{2\pi x}{3}\right)$$