Find the Fourier series for

$$f(x) = \begin{cases} 1+\dfrac{2x}{\pi}, & -\pi < x < 0 \\ 1-\dfrac{2x}{\pi}, & 0 < x < \pi \end{cases}$$

Consider $\phi_1(x) = 1+\dfrac{2x}{\pi}$

$$\phi_2(x) = 1-\frac{2x}{\pi}$$

$$\phi_1(-x) = 1+\frac{2(-x)}{\pi} = 1-\frac{2x}{\pi} = \phi_2(x)$$

$$\phi_2(-x) = 1-\frac{2(-x)}{\pi} = 1+\frac{2x}{\pi} = \phi_1(x)$$

$\therefore$ $f(x)$ is even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x-\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi-\frac{\pi^2}{\pi}\right] = \frac{2}{\pi}(\pi-\pi) = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$u = 1 - \frac{2x}{\pi}$ $\quad\quad V = \cos nx$

$u' = -2/\pi$ $\quad\quad V_1 = \frac{\sin nx}{n}$

$u'' = 0$ $\quad\quad V_2 = \frac{-\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{\pi}\frac{\cos n\pi}{n^2} - 0 + \frac{2}{\pi}\frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{\pi n^2}\left(-(-1)^n + 1\right)\right]$$

$$a_n = \frac{4}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

$\therefore$ The Fourier series is

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]\cos nx$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left[1-(-1)^n\right]\cos nx.$$

Find the Fourier series for

$$f(x) = \begin{cases} L+x & , -L < x < 0 \\ L-x & , 0 < x < L \end{cases}$$

$\phi_1(x) = L+x$

$\phi_1(-x) = L-x = \phi_2(x)$

$\phi_2(x) = L-x$

$\phi_2(-x) = L+x = \phi_1(x)$

$\therefore$ $f(x)$ is even function.

$\therefore$ The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$= \frac{2}{L}\int_{0}^{L} (L-x)\,dx$$

$$= \frac{2}{L}\left[Lx - \frac{x^2}{2}\right]_0^L = \frac{2}{L}\left[L^2 - \frac{L^2}{2}\right]$$

$$= \frac{2}{L}\left[\frac{L^2}{2}\right] = L$$

$$\boxed{a_0 = L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_{0}^{L} (L-x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = L - x$ $\qquad v = \cos\left(\frac{n\pi x}{L}\right)$

$u' = -1$ $\qquad v_1 = \frac{\sin(n\pi x/L)}{(n\pi/L)}$

$u'' = 0$

$v_2 = \frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}$

$$= \frac{2}{L}\left[(L-x)\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} - (-1)\frac{\left(-\cos\left(\frac{n\pi x}{L}\right)\right)}{\frac{n^2\pi^2}{L^2}}\right]_0^L$$

$$= \frac{2}{L}\left[0 - \frac{\cos\left(\frac{n\pi L}{L}\right)}{\frac{n^2\pi^2}{L^2}} - 0 + \frac{\cos 0}{\frac{n^2\pi^2}{L^2}}\right]$$

$$= \frac{2}{L}\cdot\frac{L^2}{\pi^2 n^2}\left[1 - \cos n\pi\right]$$

$$= \frac{2L}{n^2\pi^2}\left[-(-1)^n + 1\right]$$

$$\therefore f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{L}\right)$$

1. Find the fourier series to represent the function $f(x) = |\sin x|$, $-\pi < x < \pi$

Here $\sin x$ is an odd function. But $|\sin x|$ is an even function.

$\therefore$ The fourier Coefficient $b_n = 0$.

$\therefore$ The Fourier series for $f(x) = |\sin x|$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cdot \cdot \rightarrow ①.$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \quad [\because f(x) \text{ is even}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}[1+1] = \frac{4}{\pi}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cancel{dx}\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} |\sin x| \cos nx\,dx$$

$[\because (0,\pi), |\sin x| = \sin x]$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\{\sin(n+1)x - \sin(n-1)x] \, dx$$

$[2\cos A \sin B = \sin(A+B) - \sin(A-B)]$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \frac{1}{\pi}\left[-\frac{\cos n\pi \cos\pi}{n+1} + \frac{\cos n\pi \cos\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$[\because \sin n\pi = 0$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$[\cos\pi = -1$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi - (n+1)\cos n\pi + n-1-n-1}{n^2-1}\right]$$

$$= \frac{1}{(n^2-1)\pi}[-2\cos n\pi - 2]$$

$$= \frac{-2}{(n^2-1)\pi}[1+(-1)^n]$$

asked n is odd

$$\therefore a_n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ \dfrac{-4}{\pi(n^2-1)} & \text{when } n \text{ is even}, n \neq 1 \end{cases}$$

When $n=1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-1+1\right] = 0.$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{\pi} + \sum_{n=2,4..}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx.$$

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\sum_{n=2,4}^{\infty} \frac{\cos nx}{n^2-1}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \dots\right]$$