General Fourier Series

If $f(x)$ is a periodic function and satisfies Dirichlet's condition defined for the interval $[c, c+2l]$ then it can be represented by an infinite series is called Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n and b_n are called Fourier coefficients.

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

1. Find the Fourier series for the function $f(x) = x^2$ in $(0, 2\pi)$

$f(x) = x^2$ in $(0, 2\pi)$

Fourier series for the function $f(x)$ in $[0, 2\pi]$ is

[Put $l = \pi$]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

TO find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\,dx \qquad = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - 0\right] \qquad = \frac{1}{\cancel{\pi}}\left(\frac{8\pi^{\cancel{3}}}{3}\right)$$

$$= \frac{8\pi^2}{3} \qquad \boxed{a_0 = \frac{8\pi^2}{3}}$$

TO find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$u = x^2$ $\qquad v = \cos nx$

$u' = 2x$ $\qquad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\qquad v_2 = \frac{-\cos nx}{n^2}$

$u''' = 0$ $\qquad v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[uv_1 - u'v_2 + u''v_3\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + 2(2\pi)\frac{\cos n2\pi}{n^2} - 0 - 0 - 0 + 0\right]$$

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$$= \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) \qquad = \frac{4}{n^2}.$$

$$a_n = \frac{4}{n^2}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$u = x^2$ $\quad$ $v = \sin nx$

$u' = 2x$ $\quad$ $v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad$ $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\quad$ $v_3 = \frac{\cos nx}{n^3}$

$$= \frac{1}{\pi}\left[uv_1 - u'v_2 + u''v_3\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2\cos n2\pi}{n} + 2(2\pi)\frac{\sin n2\pi}{n^2} + 2\frac{\cos n2\pi}{n^3} + 0 - 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + 0 + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n}\right]$$

$$b_n = \frac{-4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx.$$

② Find the fourier series for the function

$f(x) = \frac{(\pi - x)^2}{2}$ in $0 \leq x \leq 2\pi$

$$f(x) = \frac{(\pi - x)^2}{2}$$

Fourier series for the function $f(x)$ in the interval $[0, 2\pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

To find a_0:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{2}\,dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\pi - x)^2\,dx = \frac{1}{2\pi} \left[\frac{(\pi - x)^3}{-3} \right]_0^{2\pi}$$

$$= \frac{-1}{6\pi} \left[(\pi - 2\pi)^3 - \pi^3 \right] = \frac{-1}{6\pi} \left[(-\pi)^3 - \pi^3 \right]$$

$$= \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3}$$

To find a_n:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{2} \cos nx\,dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$u = (\pi - x)^2$ $\qquad$ $v = \cos nx$

$u' = 2(\pi - x)(-1)$ $\qquad$ $v_1 = \frac{\sin nx}{n}$

$= -2(\pi - x)$ $\qquad$ $v_2 = \frac{-\cos nx}{n^2}$

$u'' = -2(-1) = 2,\ u''' = 0$ $\qquad$ $v_3 = \frac{-\sin nx}{n^3}$

$$a_n = \frac{1}{2\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - [-2(\pi-x)]\left[\frac{-\cos nx}{n^2}\right] + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[0 - 2(-\pi)\frac{\cos n(2\pi)}{n^2} - 0 - 0 + 2(\pi)\frac{\cos 0}{n^2} + 0\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{2\pi}\left[\frac{4\pi}{n^2}\right]$$

$$\boxed{a_n = \frac{2}{n^2}}$$

To find bn:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{2}\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$u = (\pi-x)^2$ $\quad$ $v = \sin nx$

$u' = 2(\pi-x)(-1)$ $\quad$ $v_1 = \frac{-\cos nx}{n}$

$= -2(\pi-x)$ $\quad$ $v_2 = \frac{-\sin nx}{n^2}$

$u'' = -2(-1) = 2$ $\quad$ $v_3 = \frac{\cos nx}{n^2}.$

$u''' = 0$

$$b_n = \frac{1}{2\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - \left[-2(\pi-x)\left(-\frac{\sin nx}{n^2}\right)\right] + 2\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-(\pi-2\pi)^2\frac{\cos n(2\pi)}{n} - 0 + \frac{2\cos n2\pi}{n^3} + \frac{\pi^2\cos 0}{n} + 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] \qquad \boxed{b_n = 0}$$

The fourier series is

$$f(x) = \frac{\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx + 0$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx$$