Half Range Expansions

In many Engineering Problems it is required to expand a function $f(x)$ in the range $(0,\pi)$ in a Fourier series of period 2π or in the range $(0,l)$ in a Fourier Series of period $2l$.

The half range cosine series in $(0,l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Problems on Fourier cosine series

1. Obtain the Fourier expansion of $x\sin x$ as a cosine series in $(0,\pi)$ and hence deduce the value of

$$1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \cdots$$

The fourier cosine series of $f(x)$ in $(0,\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$$

$u = x \qquad v = \sin x$

$u' = 1 \qquad v_1 = -\cos x$

$u'' = 0 \qquad v_2 = -\sin x$

$$= \frac{2}{\pi}\left[-x\cos x - (1)(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi + 0 + 0 - 0\right]$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi\right] = -2\cos\pi = -2(-1)$$

$$a_0 = 2$$

To find a_n:-

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$\sin x \cos nx = \frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x \frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right] dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(1+n)x \, dx + \int_0^{\pi} \sin(1-n)x \, dx\right]$$

$u = x$ $\quad V = \sin(1+n)x$ $\qquad u = x$ $\quad V = \sin(1-n)x$

$u' = 1$ $\quad V_1 = \dfrac{-\cos(1+n)x}{1+n}$ $\qquad u' = 1$ $\quad V_1 = \dfrac{-\cos(1-n)x}{(1-n)}$

$u'' = 0$ $\quad V_2 = \dfrac{-\sin(1+n)x}{(1+n)^2}$ $\qquad u'' = 0$ $\quad V_2 = \dfrac{-\sin(1-n)x}{(1-n)^2}$

$$= \frac{1}{\pi}\left[\left[-x\frac{\cos(1+n)x}{1+n} + \frac{\sin(1+n)x}{(1+n)^2}\right]_0^\pi + \left[-x\frac{\cos(1-n)x}{1-n} + \frac{\sin(1-n)x}{(1-n)^2}\right]_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n} - \frac{\pi\cos(1-n)\pi}{1-n}\right]$$

$$= \frac{1}{\pi}(-\pi)\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n}\right]$$

$$= (-1)\left[\frac{\cos\pi\cos n\pi - \sin\pi\sin n\pi}{1+n} + \frac{\cos\pi\cos n\pi + \sin\pi\sin n\pi}{1-n}\right]$$

$$= (-1)\left[\frac{\cos\pi\cos n\pi}{1+n} + \frac{\cos\pi\cos n\pi}{1-n}\right]$$

$$= (-1)\left[\cos\pi\cos n\pi\right]\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= (-1)(-1)(-1)^n\left[\frac{1-n+1+n}{1-n^2}\right] = (-1)^n\left[\frac{2}{1-n^2}\right]$$

$a_n = \dfrac{2(-1)^n}{1-n^2}$. $\quad \dfrac{-2(-1)^n}{n^2-1}$ $\quad$ Provided $n \neq 1$

When $n=1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \frac{\sin 2x}{2} \, dx \qquad = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$u = x$ $\qquad$ $v = \sin 2x$

$u' = 1$ $\qquad$ $v_1 = \frac{-\cos 2x}{2}$

$u'' = 0$ $\qquad$ $v_2 = \frac{-\sin 2x}{4}$

$$= \frac{1}{\pi}\left[-x\frac{\cos 2x}{2} - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi \cos 2\pi}{2} + 0\right] = \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$\boxed{a_1 = -1/2}$$

$\therefore$ The Fourier cosine series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \cos nx$$

$$= 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \cos nx$$

Deduction:-

Put $x = \pi/2$

$$\frac{\pi}{2} \sin \frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)(n+1)} \cos \frac{n\pi}{2}$$

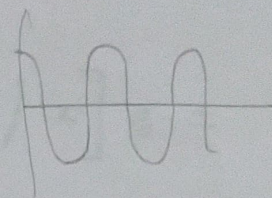
$$\frac{\pi}{2} = 1 - 0 - 2\sum_{n=2,4,\ldots}^{\infty} \frac{(-1)^n}{(n-1)(n+1)} \cos\frac{n\pi}{2}.$$

$$\left[\because \cos\frac{n\pi}{2} = 0 \quad \text{when } n \text{ is odd}\right]$$

$$\frac{\pi}{2} = 1 - 2\left[\frac{(-1)^2\cos\pi}{(2-1)(2+1)} + \frac{(-1)^4\cos 2\pi}{(4-1)(4+1)} + \cdots\right]$$

$$= 1 - 2\left[\frac{(-1)}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} \cdots\right]$$

$$\frac{\pi}{2} = 1 + 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right]$$



$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right]$$

$$\frac{\pi-2}{2} = 2\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right]$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

$$\therefore \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi-2}{4}$$

Half Range Cosine Series:

$f(x) = \underline{x(\pi - x)}$ in $0 < x < \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\pi\,dx - \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^{\pi} - \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \pi^2 - \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{3\pi^2 - 2\pi^2}{3}$$

$$= \frac{\pi^2}{3} \qquad a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(x\pi \cancel{\;x^2})\cos nx\,dx - \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= 2\int_0^{\pi} x\cos nx\,dx - \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$u = x$	$v = \cos nx$	$u = x^2$	$v = \cos nx$
$u' = 1$	$v_1 = \frac{\sin nx}{n}$	$u' = 2x$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = -\frac{\cos nx}{n^2}$	$u'' = 2$	$v_2 = \frac{-\cos nx}{n^2}$
		$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

$$= 2\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} - \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)\right.$$

$$\left. - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= 2\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] - \frac{2}{\pi}\left[\frac{2\pi \cos n\pi}{n^2} - 0\right]$$

$$= 2\left[\frac{(-1)^n - 1}{n^2}\right] - \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$= \frac{2(-1)^n}{n^2} - \frac{2}{n^2} - \frac{4(-1)^n}{n^2}$$

$$= \frac{-2(-1)^n}{n^2} - \frac{2}{n^2} = \frac{-2}{n^2}\left[(-1)^n + 1\right]$$

$$a_n = \begin{cases} -4/n^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore\ f(x) = \frac{\pi^2}{6} - 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

② Find Half Range cosine series for $f(x) = x$ in $0 < x < \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{\pi} = \pi \quad \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} -4/n^2\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2\pi} \right) \cos nx$$

$$= \frac{\pi}{2} + \frac{-4}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$