Fourier Sine Transform:

The fourier sine transform of $f(x)$ is defined by,

$$F_S[S] = F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

The inverse fourier sine transform of $F_S(S)$ is given by,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S(S)\sin Sx\,ds$$

Fourier cosine Transform:

The Fourier cosine transform of $f(x)$ is defined by

$$F_C[S] = F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos Sx\,dx$$

The inverse Fourier cosine transform of $F_C(S)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C[S]\cos Sx\,ds$$

Parseval's Identity:

Sine Transform:

If F(S) is the Fourier transform of $f(x)$, then

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_S(S)]^2 ds$$

Cosine Transform:

If F(S) is the Fourier transform of $f(x)$, then

$$\int_0^\infty [f(x)]^2 dx = \int_0^\infty [F_C(S)]^2 ds.$$

1]. Find the FST of $f(x)$ defined as

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Soln.:

$$F_S(S) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin Sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin Sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos Sx}{S}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos S}{S}\right]$$

2]. Find the FST of $1/x$.

Soln.: $F_S(S) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin Sx \, dx$

5]. Find the FCT of $\frac{e^{-ax}}{x}$ and hence, find $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

Soln. :

$$F_c[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

$$\frac{d}{ds}F_c[s] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(-x\sin sx)\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\frac{d}{ds}F_c[s] = -\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

~~$F_c(s)$~~

Integrating, we get

$$F_c[s] = -\sqrt{\frac{2}{\pi}}\int\frac{s}{s^2+a^2}\,ds$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int\frac{2s}{s^2+a^2}\,ds$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = -\frac{1}{\sqrt{2\pi}}\log(s^2+a^2)$$

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$$F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}}\log(s^2+b^2)$$

Now,

$$F_c\left[\frac{e^{ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2) + \frac{1}{\sqrt{2\pi}}\log(s^2+b^2)$$

$$= \frac{1}{\sqrt{2\pi}}\log\left[\frac{s^2+b^2}{s^2+a^2}\right]$$

Hw

1]. Show that $e^{-x^2/2}$ is self reciprocal under FCT.

2]. Find the FST of the function $f(x)=\frac{e^{-ax}}{x}$ and hence find $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

3]. Find FST of e^{-ax}, $a>0$.

4]. Find the FST of $\frac{x}{x^2+a^2}$

5]. Find the FCT of $f(x) = \begin{cases} \cos x, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$

6]. Show that $e^{-x^2/2}$ is self-reciprocal under FCT.