Convolution Theorem of Two functions

The convolution of two functions $f(x)$ and $g(x)$ is defined as,

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Convolution Theorem:

The fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their fourier transforms.

ie, $$F[f(x)*g(x)] = F(S)\cdot G(S)$$
$$= F[f(x)]\, F[g(x)]$$

Problems based on Convolution:

1. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

(or)

Find the Fourier cosine transform of $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ and Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

(or)

Evaluate parseval's Identity $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

proof: Consider $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

$$F_c[S] = F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$

$$F_c[g(x)] = F_c[S] = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}}\frac{b}{b^2+s^2}$$

we know that

$$\int_0^\infty F_c[f(x)]\, F_c[g(x)]\, ds = \int_0^\infty f(x) \cdot g(x)\, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}\, ds = \int_0^\infty e^{-ax} e^{-bx}\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{ab}{(a^2+s^2)(b^2+s^2)}\, ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2ab}{\pi} \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-1}{a+b}[0-1]$$

$$= \frac{1}{a+b}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

2]. Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$ using Transform.

Soln.: consider $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

$$F_S[f(x)] = F_S[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

$$F_S[g(x)] = F_S[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2}$$

Now,

$$\int_0^\infty F_S[f(x)]\, F_S[g(x)]\, ds = \int_0^\infty f(x) \cdot g(x)\, dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\sqrt{\frac{2}{\pi}}\frac{s}{s^2+b^2}\,ds = \int_0^\infty e^{-ax}e^{-bx}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-1}{(a+b)}\left[0-1\right]$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \frac{\pi}{2(a+b)}$$

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a+b)}$$