Half Range Expansion

Half Range Cosine Series: HRCS in $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Half Range Sine Series: HRSS in $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

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1. Find the half range cosine series for $f(x) = x$ in $0 < x < \pi$.

Soln.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{\pi^2}{\pi}$$

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$$

$u = x$, $V = \cos nx$

$u' = 1$, $V_1 = \sin nx/n$

$u'' = 0$, $V_2 = -\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$n = 1$, $(-1)^1 - 1 = -1 - 1 = -2$

$n = 2$, $(-1)^2 - 1 = 1 - 1 = 0$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{n^2}\cos nx.$$

2]. Find the half range Sine Series for $f(x) = x$ in $(0, l)$

Soln.

HRSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$$

$u = x$	$v = \sin\left(\frac{n\pi x}{l}\right)$
$u' = 1$	$v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}$
$u'' = 0$	$v_2 = \frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$$

$$= \frac{2}{l}\left[x\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right) - 1\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-x\frac{\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l} + \frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi} l \cos n\pi + 0\right]$$

$$b_n = -\frac{2l}{n\pi}(-1)^n$$

$$\therefore f(x) = \frac{-2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{l}\right)$$

5). Express $f(x) = x(\pi - x)$; $0 < x < \pi$ as a Fourier series of periodicity 2π containing i). Sine terms only ii). cosine terms only.

Soln.

HRSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$u = \pi x - x^2$	$v = \sin nx$
$u' = \pi - 2x$	$v_1 = -\frac{\cos nx}{n}$
$u'' = -2$	$v_2 = -\frac{\sin nx}{n^2}$
$u''' = 0$	$v_3 = \frac{\cos nx}{n^3}$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(\pi x - x^2)\left(\frac{\cos nx}{n}\right) + (\pi - 2x)\left(\frac{\sin nx}{n^2}\right) - 2\,\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}\cos n\pi - \left(-\frac{2}{n^3}\cos 0\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}(-1)^n + \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}\,\frac{2}{n^3}\left[1 - (-1)^n\right]$$

$$b_n = \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} \frac{8}{n^3\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$n = 1$, $1 - (-1)^1$
$1 + 1 = 2$

$n = 2$; $1 - (-1)^2$
$1 - 1 = 0$

$$\therefore f(x) = \sum_{n=odd}^{\infty} \frac{8}{n^3\pi} \sin nx$$

HRCS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - 0\right]$$

$$= \frac{2}{\pi}\pi^3\left(\frac{3-2}{6}\right)$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx$$

$u = \pi x - x^2$	$v = \cos nx$
$u' = \pi - 2x$	$v_1 = \sin nx/n$
$u'' = -2$	$v_2 = -\cos nx/n^2$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[+(\pi - 2\pi)\frac{\cos n\pi}{n^2} - \left(+(\pi - 0)\frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n^2} - \frac{\pi}{n^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{-\pi}{n^2}\right)\left[(-1)^n + 1\right]$$

$$= \frac{-2}{n^2}\left[1 + (-1)^n\right] = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{-4}{n^2}, & \text{if } n \text{ is even} \end{cases}$$

HRCS

$$f(x) = \frac{\pi^2/3}{2} + \sum_{n=even}^{\infty} \frac{-4}{n^2}\cos nx$$

$$= \frac{\pi^2}{6} - 4\sum_{n=even}^{\infty} \frac{1}{n^2}\cos nx$$